Quotient graphs for certain arithmetic subgroups of PGL_3 over a rational function field

Dissertation

zur Erlangung des akademischen Grades doctor rerum naturalium

vorgelegt von

Herrn Bastian Christ
geb. am 27. 06. 1988 in Gießen

am

Mathematischen Institut
der Justus-Liebig-Universität Gießen

April 2018

Betreuer: Prof. Dr. Ralf Köhl

Contents

Deutsche Zusammenfassung

Das Ziel der vorliegenden Arbeit ist es die Ergebnisse der Arbeit [KMS15] zu verallgemeinern. Wir betrachten die Wirkung arithmetischer Untergruppen der projektiven Gruppe über einem rationalen Funktionenkörper auf dem zugehörigen Bruhat-Tits Gebäude. Für die Einschränkung dieser Wirkung auf den zugrundeliegenden Graphen des Bruhat-Tits Gebäudes werden wir die jeweiligen Quotientengraphen bestimmen. Auf dem rationalen Funktionenkörper über einem endlichen Körper $\mathbb{F}_q$ mit q Elementen können wir diskrete Bewertungen definieren. Diese Bewertungen korrespondieren zu normierten, irreduziblen Polynomen über $\mathbb{F}_q$. Der Grad der Bewertung ist gleich dem Grad des irreduziblen Polynoms, welches zu dieser Bewertung assoziiert ist. Für kleine Grade ist der berechnete Quotientengraph ein Fundamentalbereich für die Wirkung der arithmetischen Untergruppe auf dem zugrundeliegenden Graphen des zugehörigen Bruhat-Tits Gebäudes. Ein solcher Fundamentalbereich kann beispielsweise dazu verwendet werden eine Präsentierung der Gruppe zu bestimmen. Dazu werden nur noch die Stabilisatoren innerhalb der arithmetischen Untergruppe von bestimmten Ecken und Kanten benötigt (Siehe [Ser80], Abschnitt I.4 und Abschnitt II.2). Einige dieser Stabilisatoren werden wir in der vorliegenden Arbeit bestimmen, aber wir werden nicht genauer darauf eingehen und uns auf die Berechnung der Quotientengraphen konzentrieren. Zu einer Stelle p des rationalen Funktionenkörpers berechnen wir den Quotientengraphen der Wirkung der Gruppe $\mathrm{PGL}_3(\mathcal{O}_{\{p\}})$ auf dem zugrundeliegenden Graphen X des zugehörigen Bruhat-Tits Gebäudes. Hierbei bezeichnet $\mathcal{O}_{\{p\}}$ alle Elemente des rationalen Funktionenkörpers, die nur an der Stelle p mögliche Polstellen besitzen. Zuerst berechnen wir den Fundamentalbereich für die Wirkung der $\mathrm{PGL}_3(\mathcal{O}_{\{\infty\}}) \cong \mathrm{PGL}_3(k[t])$ auf dem zugehörigen Graphen. Um die Quotientengraphen für alle Stellen p berechnen zu können betrachten wir die Wirkung von $\mathrm{PGL}_3(\mathcal{O}_{\{p,\infty\}})$ auf dem Produkt der Bruhat-Tits Gebäude, die zu den Stellen p und ∞ assoziiert sind. Mit $\mathcal{O}_{\{p,\infty\}}$ bezeichnen wir die Funktionen, die nur Polstellen in p und ∞ haben können. Indem wir die Orbits der Wirkung der $\mathrm{PGL}_3(\mathcal{O}_{\{p\}})$ auf den Ecken des zu p assoziierten Bruhat-Tits Gebäudes mit Orbits der Wirkung von $\mathrm{PGL}_3(\mathcal{O}_{\{\infty\}})$ auf den Ecken des zu ∞ assoziierten Bruhat-Tits Gebäudes identifizieren finden wir eine Beschreibung der Ecken des Quotientengraphen $\mathrm{PGL}_3(\mathcal{O}_{\{p\}})\backslash X$. Die Anzahl an Kanten zwischen zwei vorgegebenen Ecken des Quotientengraphen $\mathrm{PGL}_3(\mathcal{O}_{\{p\}})\backslash X$ kann über die Anzahl von geeigneten Doppelnebenklassen bestimmt werden. Um die Anzahl an Doppelnebenklassen bestimmen zu können müssen wir einige Fälle unterscheiden. Für jeden dieser Fälle benötigen wir die Mächtigkeiten der involvierten Mengen um damit die Anzahl der Doppelnebenklassen in dem jeweiligen Fall zu berechnen. Das Hauptresultat der vorliegenden Arbeit beschreibt die Quotiengraphen durch Angabe der Anzahl an Kanten zwischen zwei vorgegebenen Ecken.

Introduction

In order to study semisimple complex Lie groups and semisimple algebraic groups over an arbitrary field Tits introduced buildings (see [Tit74]). So he associated to such a group a building, such that the group acts in a nice way on the associated building. By studying this action one can get information about the group. For instance, when we have a fundamental domain for an action of a group on a graph together with the stabilizers we can derive a presentation of the group by generators and relations (cf. [Ser80], Sections I.4, II.2). More general, if we have a simplicial fundamental domain for the action of a group on a certain simplicial complex together with the stabilizers we can get a presentation of the group by generators and relations (see [Sou79], Section 2 or [AB08], Chapter 14). For the connection between buildings and groups Tits introduced the theory of BN-pairs. When we can find a BN-pair in a group G, which means that B and N are subgroups of G satisfying certain properties, then we may associate a building to G with a strongly transitive action of G on that building. If the corresponding Weyl group, defined by $W = N/(B \cap N)$, is a finite reflection group, then the associated building is called a spherical building. In this case the apartments are triangulated spheres. For many groups with a BN-pair such that the associated building is spherical, it is possible to assign a second BN-pair to the group whenever we can equip the ground field with a discrete valuation. Then the Weyl group for this second BN-pair is an infinite Euclidean reflection group and an apartment of the associated building is a triangulated Euclidean space. They are called affine or Euclidean buildings. In [IM65] Iwahori and Matsumoto introduced first such BN-pairs and later in [BT72] Bruhat and Tits generalized this result to construct such BN-pairs for reductive algebraic groups over a non-archimedean local field. According to this construction, Euclidean buildings are often called Bruhat-Tits buildings. There exist some different compactifications for Bruhat-Tits buildings (see, for example, [Lan96], [Wer01], [Wer04] and [Wer07]). In [Wei09] there is a classification of Bruhat-Tits buildings. Here a Bruhat-Tits building is a thick, irreducible, affine building whose building at infinity is Moufang. Bruhat-Tits buildings play an important role by finiteness properties of matrix groups in positive characteristic. For example, Serre proved that $\mathrm{SL}_2(\mathbb{F}_q[t])$ is not finitely generated (which is a theorem of Nagao) by analyzing the stabilizers corresponding to a fundamental domain for the action of $\mathrm{SL}_2(\mathbb{F}_q[t])$ on the Bruhat-Tits tree (see [Ser80], Section II.1.6). Behr gave in [Beh98] a characterization for arithmetic subgroups to be finitely generated and finitely presented. More general the finiteness length of such matrix groups was studied (see, for instance, [Abr96], Section III.2, [AB08] Section 13.5, [Wit14] or [BKW13]).

Serre studied in [Ser80] the action of the projective linear group of the two dimensional vector space over a global function field of a smooth projective curve over a finite field. For such a global function field we can define a discrete valuation and associate a Bruhat-Tits building to the projective linear group over the two dimensional vector space over this field. In that case the Bruhat-Tits building is a graph and in particular it is a tree. Therefore the Bruhat-Tits building is called Bruhat-Tits tree in this case. Serre computed fundamental domains for the action of arithmetic subgroups on the Bruhat-Tits tree for some degrees of the valuations. In the case that the global function field is the rational function field over a finite field $\mathbb{F}_q$ with q elements this result was generalized by Köhl, Mühlherr and Struyve in [KMS15] to arbitrary degrees. But for higher degree the orbit space $\Gamma\backslash X$, where X is the Bruhat-Tits tree and Γ is the arithmetic subgroup acting on X, is not a fundamental domain for this action. There is another generalization of the results by Serre in [Sou79]. There Soulè computed the fundamental domain for the action of $G(k[t])$, where G is a simple and simply-connected Chevalley group defined over $\mathbb{Z}$ and $k[t]$ denotes the ring of polynomials over a field k.

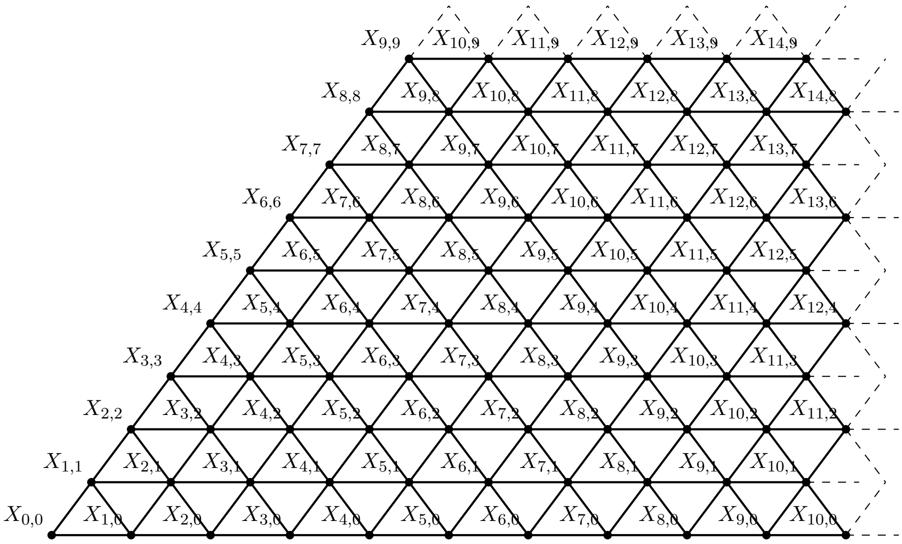
In the present thesis we want to generalize the result in [KMS15] of arithmetic subgroups of $\mathrm{PGL}_2(k(t))$ to arithmetic subgroups of $\mathrm{PGL}_3(k(t))$. The associated building to $\mathrm{PGL}_3(k(t))$ is a Bruhat-Tits building of type $\tilde{A}_2$, so the apartments in this affine building are Euclidean planes tiled by equilateral triangles. We consider only the underlying graph, i.e. the 1-skeleton, of the Bruhat-Tits building and the goal of this Thesis is to describe the orbit space $\Gamma\backslash X$, where X is this underlying graph of the Bruhat-Tits building and Γ is an arithmetic subgroup of $\mathrm{PGL}_3(k(t))$. The Main Theorem 3.1.1 describes the vertices of this quotient and the number of edges between two given vertices in the quotient graph. To reach this goal we adopt the strategy from [KMS15].

In Chapter 1 we summarize the needed definitions and some important facts about buildings. To do so we mostly follow the book by Abramenko and Brown [AB08]. First we introduce simplicial complexes, because we want to define buildings as simplicial complexes that can be covered in a certain way by subcomplexes, the so called apartments. We consider chamber complexes, which are certain simplicial complexes. They are finite dimensional and the simplices with maximal dimension are called chambers. Buildings are examples of chamber complexes. Then we consider finite reflection groups and Coxeter groups. To this groups we can associate a simplicial complex which is isomorphic to the apartments of the building. The simplicial complexes corresponding to Coxeter groups are called Coxeter complexes. When the Coxeter complex is finite, then the building is called spherical. If the geometric realization of the Coxeter complex is a Euclidean space, then the building is called affine or Euclidean. Moreover, we consider the connection between buildings and groups with a BN-pair as mentioned above. In particular we assign a building to the group $\mathrm{SL}_n(k)$ over some field k. To certain affine buildings we can associate a building at infinity, which is a spherical building.

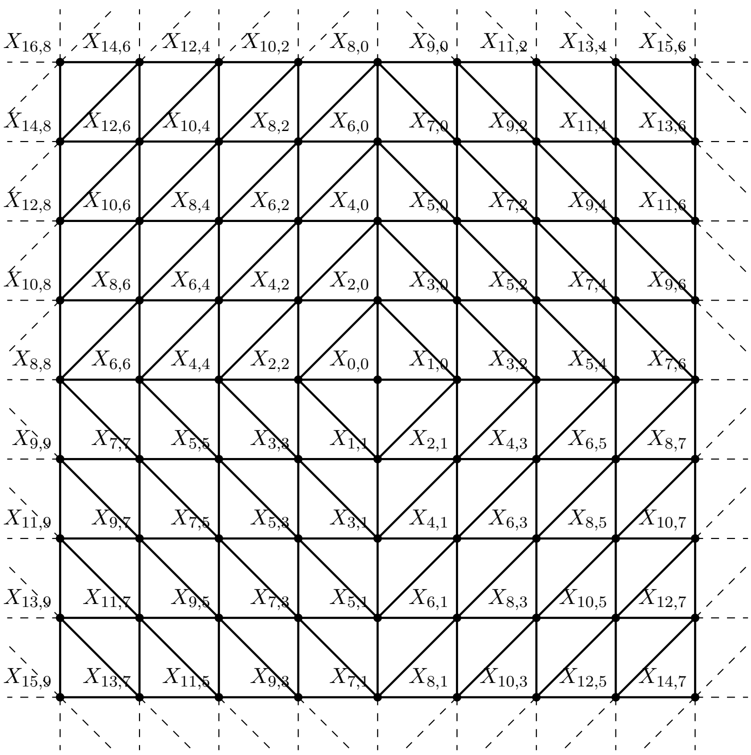
In Chapter 2 we define the building associated to the group $\mathrm{PGL}_n(K_\nu)$, where K is a field with discrete valuation ν. We use lattice classes in order to describe that building, in particular the vertices of the building are given by the lattice classes of the n-dimensional vector space over K. This building is isomorphic to the affine building for $\mathrm{SL}_n(K)$, defined by a BN-pair for certain subgroups B and N of $\mathrm{SL}_n(K)$. Now the latter building is a Bruhat-Tits building and hence the former is a Bruhat-Tits building, too. A Bruhat-Tits building is an affine building whose building at infinity is Moufang. For $\mathrm{SL}_n(K)$ the associated affine building has as building at infinity the associated spherical building for $\mathrm{SL}_n(K)$. Due to a result by Tits (announced in [Tit74], p.274) every thick, irreducible, spherical building of rank at least 3 is Moufang (See [AB08], Theorem 7.59). For the spherical building of $\mathrm{SL}_n(K)$ it is shown in [AB08], Section 7.3.4 that it satisfies the Moufang property. Then we restrict to the case of the Bruhat-Tits building of rank 3 associated to $\mathrm{PGL}_3(K_\nu)$, where $K = k(t)$ is the rational function field over a finite field $k = \mathbb{F}_q$ with q elements. In this case we generalize some facts from [Ser80]. We consider the action of the arithmetic subgroup $\mathrm{PGL}_3(k[t])$ on the underlying graph of the corresponding Bruhat-Tits building. For this action we compute the fundamental domain and some stabilizers.

In Chapter 3 we generalize the result in [KMS15] to $\mathrm{PGL}_3(K_\nu)$, where $K = k(t)$ is the rational function field over a finite field $k = \mathbb{F}_q$ with q elements. In particular we calculate the quotient graph of the underlying graph of the associated Bruhat-Tits building by the action of an arithmetic subgroup of $\mathrm{PGL}_3(K_\nu)$. Similar to [KMS15] we consider for a place p of $k(t)$ and the place ∞ the action of $\mathrm{PGL}_3(\mathcal{O}_{\{p,\infty\}})$, where $\mathcal{O}_{\{p,\infty\}}$ denotes the elements of $k(t)$ having only possible poles at p or ∞, on the product of the Bruhat-Tits building associated to p and the Bruhat-Tits building associated to ∞. Using the fundamental domain for the action of $\mathrm{PGL}_3(\mathcal{O}_{\{\infty\}}) \cong \mathrm{PGL}_3(k[t])$ on the underlying graph of the Bruhat-Tits building corresponding to the place ∞ we find the vertices of the quotient $\Gamma\backslash X$, where X is the underlying graph of the Bruhat-Tits building corresponding to the place p and $\Gamma = \mathrm{PGL}_3(\mathcal{O}_{\{p\}})$; here $\mathcal{O}_{\{p\}}$ denotes the elements in $k(t)$ having only possibly poles at p. For the edges of the quotient graph we have to compute the size of certain double cosets. Therefore we have to distinguish some cases. When we compute the length of the double cosets in each of these cases we find the number of edges between two given vertices in the quotient graph. So we prove our Main Theorem 3.1.1. Since it needs 9 pages to formulate the main result of this thesis we will not state it again in this introduction. But here are the pictures of the quotient graphs for the degrees $d \in \{1, 2, 3\}$:

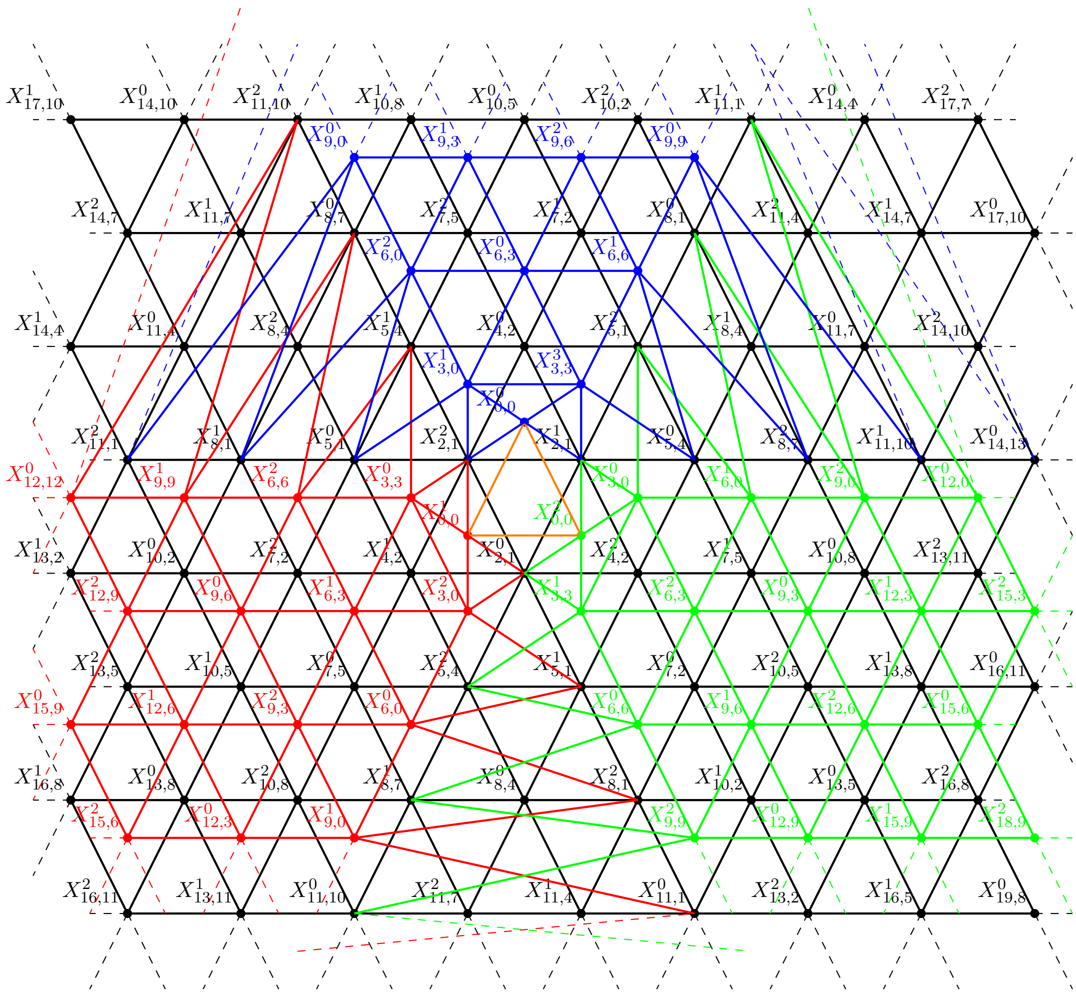
For degree 1:



For degree 2:



For degree 3:



Acknowledgements.

Without the help of some people it would not be possible to write the present thesis. First of all I want to thank my advisor Ralf Köhl for his great support. It was a pleasure to work with him and he always had an open door for questions. His enthusiasm was very motivating and he helped me to find the right direction if necessary.

I would also like to thank Bernhard Mühlherr and Max Horn for some constructive and helpful discussions.

Further I want to thank my colleagues in the math institute. It was a pleasant atmosphere and I enjoyed to work with all the nice people there. Here I want to mention some people I worked with in context of my teaching assistant position during the last five years: Ralf Köhl, Björn Walther, Max Horn, Robert Zeise, Paula Harring, Julius Grüning, Klaus Metsch, Erich Häusler, Frank Lamping, Christian Weigel and Thomas Meixner. Moreover, I want to thank all people in the AG Algebra for the great time I had during these last years.

Of course I would like to thank my friends and my family. Especially I want to thank my parents who always supported and helped me.

1. Buildings

In the first chapter we summarize definitions and important facts about buildings. We will mainly follow the book [AB08]. There exist several approaches to buildings. We define buildings as simplicial complexes, that can be covered by Coxeter complexes.

1.1. Simplicial complexes

This section is based on Chapter 3 in [Pap12] and [AB08], Appendix A.1.1.

Definition 1.1.1. Let $\mathcal{V}$ be a set. An *abstract simplicial complex* $\mathcal{K}$ with vertex set $\mathcal{V}$ is a nonempty collection of finite subsets of $\mathcal{V}$ such that:

- if $v \in \mathcal{V}$, then $\{v\} \in \mathcal{K}$
- if $\sigma \in \mathcal{K}$ and $\tau \subset \sigma$, then $\tau \in \mathcal{K}$

An element $v \in \mathcal{V}$ is called *vertex*. An element $\sigma \in \mathcal{K}$ is called *simplex* of $\mathcal{K}$. If the simplex σ consists of $k+1$ vertices, then we call it a *k-simplex* of $\mathcal{K}$. Furthermore the cardinality $|\sigma| = k+1$ is called the *rank of the simplex* and k is the *dimension of the simplex* σ. The empty set is a simplex of rank 0 and dimension -1. For a vertex $v \in \mathcal{V}$ we also call the corresponding 0-simplex $\{v\}$ of $\mathcal{K}$ a *vertex of* $\mathcal{K}$. If e is a 1-simplex of $\mathcal{K}$ we call it an *edge of* $\mathcal{K}$.

Remark 1.1.2. We say simplicial complex for an abstract simplicial complex.

Remark 1.1.3. Another way to define simplicial complexes is the following: Let Δ be a non-empty poset (partial ordered set). Then Δ is a simplicial complex if it satisfies the following conditions:

1. Any two elements $A, B \in \Delta$ have a greatest lower bound $A \cap B$.
2. For any $A \in \Delta$, the poset $\Delta_{\leq A}$ of faces of A is isomorphic to the poset of subsets of $\{1, ..., r\}$ for some integer $r \geq 0$.

Definition 1.1.4. Let $\mathcal{K}$ be a simplicial complex and L be a subcollection of $\mathcal{K}$. Then L is a *subcomplex* of $\mathcal{K}$ if each subset of an element in L is itself contained in L.

Remark 1.1.5. If L is a subcomplex of $\mathcal{K}$, then it is itself a simplicial complex and its set of vertices is a subset of the set of vertices of $\mathcal{K}$.

Definition 1.1.6. Let $\mathcal{K}$ be a simplicial complex. For all $n \in \mathbb{N}_0$ the *n-skeleton* $\mathcal{K}_n$ of $\mathcal{K}$ is the subcomplex of $\mathcal{K}$ consisting of all k-simplices with $k \leq n$. Its vertex set is the union of all the k-simplices of $\mathcal{K}$ with $k \leq n$.

Definition 1.1.7. If $\tau \subset \sigma \in \mathcal{K}$, then τ is called a *face of the simplex* σ. A *maximal simplex* of $\mathcal{K}$ is a simplex which is not a proper face of any simplex of $\mathcal{K}$.

Definition 1.1.8. The simplicial complex $\mathcal{K}$ has *finite dimension* $n \in \mathbb{N}$, if all simplices in $\mathcal{K}$ are k-simplices for some $k \leq n$ and n is the minimal natural number with this property. Then we also say that $\mathcal{K}$ has *rank* $n+1$.

Definition 1.1.9. A simplicial complex of dimension 1 is a (simplicial) *graph*. In particular, a graph is a pair (V, E) of sets, where V is the vertex set of the simplicial complex and E is the set of edges.

Remark 1.1.10. If it is necessary we can consider a graph as a directed graph (which means that the set of edges E consists of ordered subsets of $V \times V$) by choosing for each edge a direction. For more details about directed graphs see for example [Ser80] Chapter 1, Section 2.1.

Definition 1.1.11. If we have an edge (u, v) in some directed graph we call u the *origin of the edge*.

Definition 1.1.12. The 1-skeleton of a simplicial complex $\mathcal{K}$ is a graph and it is called the *underlying graph* of $\mathcal{K}$.

Definition 1.1.13. Let $\mathcal{K}$ be a simplicial complex with vertex set $\mathcal{V}$ and L be a simplicial complex with vertex set W. A *simplicial map* from $\mathcal{K}$ to L is a map $\varphi : \mathcal{V} \to W$ such that for each simplex $\sigma \in \mathcal{K}$ the image $\varphi(\sigma)$ is a simplex in L. If there exist a simplicial map ψ from L to $\mathcal{K}$, such that φ and ψ are inverse functions, then φ is a *simplicial isomorphism*. Furthermore, if $L = \mathcal{K}$ and φ is an isomorphism from $\mathcal{K}$ to itself, we call φ a *simplicial automorphism* of $\mathcal{K}$. Write $\mathrm{Aut}(\mathcal{K})$ for the set of all automorphisms of $\mathcal{K}$.

1.2. Geometric realization

Based on [AB08] Appendix A.1.1.

Definition 1.2.1. Let Δ be a simplicial complex with vertex set $\mathcal{V}$. Further, let V be a real vector space with $\mathcal{V}$ as a basis. For each nonempty simplex $A \in \Delta$ set

$$|A| = \left\{ \sum_{v \in A} \lambda_v v \mid \lambda_v > 0 \text{ for all } v \in A \text{ and } \sum_{v \in A} \lambda_v = 1 \right\}$$

and define

$$|\Delta| = \bigcup_{A \in \Delta} |A|.$$

Then $|\Delta|$ is called the *geometric realization* of Δ.

Remark 1.2.2. We can define a topology on $|\Delta|$ as follows: Consider closed simplices

$$|\bar{A}| := \left\{ \sum_{v \in A} \lambda_v v \mid \lambda_v \geq 0 \text{ for all } v \in A \text{ and } \sum_{v \in A} \lambda_v = 1 \right\} = \bigcup_{B \leq A} |B|$$

for each $A \in \Delta$. If the rank of A is r, then we can visualize the closed simplex $|\bar{A}|$ as the convex hull in $\mathbb{R}^r$ of its r vertices. So we define a topology on each closed simplex as a subspace of Euclidean space. Then the topology on $|\Delta|$ is defined by calling a subset closed if and only if its intersection with each closed simplex is closed.

1.3. Flag complexes

The definition is taken from [AB08] Appendix A.1.2. and the example can be found in [AB08] Section 4.3.

Definition 1.3.1. Let P be a set with a binary incidence relation, which is reflexive and symmetric. A *flag* in P is a set of pairwise incident elements of P. The *flag complex* associated to P is the simplicial complex $\Delta(P)$ with P as vertex set and the finite flags as simplices. A flag complex of dimension 1 is called an *incidence graph*.

Example 1.3.2. Let V be a vector space over an arbitrary field k of finite dimension $n \geq 2$. Consider the projective space $\mathbb{P}_{n-1}(V)$ consisting of all nonzero proper subspaces of V. We define an incidence relation on the projective space by calling two subspaces A, B of V incident if $A \subseteq B$ or $B \subseteq A$. Then the flags in $\mathbb{P}_{n-1}(V)$ are the chains

$$\{V_1 \subseteq ... \subseteq V_r \mid V_i \text{ is a subspace of } V \text{ with } dim_k V_i = i \text{ for all } 1 \leq i \leq r\}$$

for some integer $1 \leq r \leq n$. These are the simplices of the flag complex corresponding to the vector space V.

1.4. Chamber complex

The definitions can be found in [Bro96] Chapter I Appendix C and in [AB08] Appendix A.1.3.

Definition 1.4.1. Let $n \in \mathbb{N}$ and Δ a n-dimensional simplicial complex. A *gallery* is a sequence of n-simplices $\Gamma = (C_0, ..., C_d)$ such that for all $i \in \{1, ..., d\}$ we have $C_{i-1} = C_i$ or $C_{i-1} \cap C_i$ is a $(n-1)$-simplex of Δ. In the latter case we say that C_{i-1} and C_i are *adjacent*. If there exist a gallery $\Gamma = (C_0, ..., C_d)$ from C_0 to C_d, then we say that they are *connected* by Γ. The integer d is the *length* of Γ.

Definition 1.4.2. Let $n \in \mathbb{N}$ and Δ be a n-dimensional simplicial complex. Then Δ is a *chamber complex* if it satisfies the two conditions:

- all maximal simplices of Δ are n-simplices
- any two maximal simplices of Δ can be connected by a gallery

The maximal simplices are called *chambers*. The $(n-1)$-simplices will be called *panels*. We say that a chamber complex is *thin* if each panel is a face of exactly two chambers.

Definition 1.4.3. Let Δ be a chamber complex. On the set $\mathcal{C}(\Delta)$ of chambers we can define a *distance function* d by

$$d(C, D) = min\{r \mid \text{ there exists a gallery of length } r \text{ from } C \text{ to } D\}$$

for chambers $C, D \in \mathcal{C}(\Delta)$.

Definition 1.4.4. A gallery from a chamber C to a chamber D is called *minimal gallery* if the length of the gallery is equal to $d(C, D)$.

Definition 1.4.5. Let Δ be a chamber complex of dimension $n-1$, with $n \in \mathbb{N}$, and let I be a set with n elements. A *type function* on Δ with values in I is a function τ that assigns to each vertex v an element $\tau(v) \in I$ such that the vertices of every chamber are mapped bijectively onto I. Is v a vertex of Δ then we call $\tau(v)$ the *type* of v. Moreover, if A is a simplex with vertices $v_1, ..., v_k$ then we call $\tau(A) = \{\tau(v_1), ..., \tau(v_k)\}$ the *type* of A. The *cotype* of such a simplex A is defined to be the complement $I \setminus \tau(A)$. The *codimension* of A is defined to be $|I \setminus \tau(A)|$. If we have a panel of cotype $\{i\}$ for some $i \in I$, then we call it an *i-panel*.

Definition 1.4.6. We say that Δ is *colorable* if it admits a type function.

Definition 1.4.7. A simplicial map φ which maps chambers to chambers is called *chamber map*.

Remark 1.4.8. A chamber map takes galleries to galleries.

Definition 1.4.9. A *chamber subcomplex* of a chamber complex Δ is a simplicial subcomplex Σ that is a chamber complex in its own right and has the same dimension as Δ.

Remark 1.4.10. The image of a chamber map $\Delta \to \Delta'$ is always a chamber subcomplex of Δ'. If Σ is a chamber subcomplex of Δ, then the inclusion $\Sigma \hookrightarrow \Delta$ is a chamber map.

Definition 1.4.11. Let Δ be a chamber complex. The *link of a simplex* A is the subcomplex

$$lk_\Delta(A) = \{B \in \Delta \mid A \cap B = \emptyset, A \cup B \in \Delta\}.$$

1.5. Chamber system

From [AB08] Appendix A.1.4.

Definition 1.5.1. Let Δ be a chamber complex which is colorable. So there is a type function with values in a set I. Then any panel of Δ has cotype $\{i\}$ for some $i \in I$, in particular it is an i-panel. For $i \in I$ we say that two adjacent chambers of Δ are *i-adjacent* if their common panel is an i-panel. The *chamber system* associated to Δ is the set $\mathcal{C} = \mathcal{C}(\Delta)$ of chambers together with the relations of i-adjacency, one for each $i \in I$. For a subset $J \subseteq I$ we call two chambers *J-equivalent* if they can be connected by a gallery $C_1, ..., C_k$, for some $k \in \mathbb{N}$, such that any two consecutive chambers C_{r-1}, C_r are j-adjacent for some $j \in J$. The equivalence classes of chambers under J-equivalence are called *J-residues*, or residues of type J.

1.6. Finite reflection groups

This section follows [AB08] Chapter 1.

Let V be a *Euclidean vector space*, i.e. a finite-dimensional real vector space with an inner product.

Definition 1.6.1. A *hyperplane* H in V is a subspace of codimension 1. The *reflection* with respect to H is the linear tansformation $s_H : V \to V$ that is the identity on H and is multiplication by -1 on the 1-dimensional orthogonal complement of H.

Definition 1.6.2. A *finite reflection group* is a finite group W of invertible linear transformations of V generated by reflections s_H, where H ranges over a set $\mathcal{H}$ of hyperplanes. $\mathcal{H}$ is said to be a *hyperplane arrangement.*

Definition 1.6.3. A reflection group W is called *essential* if $\bigcap\limits_{H \in \mathcal{H}} H = 0$. The *rank of a finite reflection group* W is the dimension of the orthogonal complement of $\bigcap\limits_{H \in \mathcal{H}} H$ in V.

Definition 1.6.4. A finite reflection group W is called *reducible* if it decomposes into a direct product $W = W_1 \times W_2$ of finite reflection groups W_1, W_2 of the summands of $V = V_1 \oplus V_2$ respectively, where V_1 and V_2 are both nontrivial. Otherwise W is called *irreducible.*

Remark 1.6.5. Let $\mathcal{H} = \{H_i\}_{i \in I}$ for some finite set I be a hyperplane arrangement. For each $i \in I$ choose a nonzero linear function $f_i : V \to \mathbb{R}$ such that H_i is defined by $f_i = 0$. Then the function f_i is uniquely determined by the hyperplane H_i, up to multiplication by a nonzero scalar.

Definition 1.6.6. A *cell* in V with respect to $\mathcal{H}$ is a nonempty set A obtained by choosing for each $i \in I$ a sign $\sigma_i \in \{+, -, 0\}$ and specifying $f_i = \sigma_i$. [Here "$f_i = +$" means $f_i > 0$ and "$f_i = -$" means $f_i < 0$] Hence we have $A = \bigcap_{i \in I} U_i$, where U_i is either H_i or one of the open half-spaces of V determined by H_i. The sequence $\sigma := (\sigma_i)_{i \in I}$ that encodes the definition of A is called the *sign sequence* of A and is denoted by $\sigma(A)$. The *support* of A is the subspace $\operatorname{supp}(A) = \bigcap_{\sigma_i(A)=0} H_i$. The *dimension of a cell* A is defined to be the dimension of its support $\operatorname{supp}(A)$. We denote by $\Sigma(\mathcal{H})$ the set of cells.

Definition 1.6.7. Given cells $A, B \in \Sigma(\mathcal{H})$, we call B a *face* of A and write $B \leq A$ if for each $i \in I$ either $\sigma_i(B) = 0$ or $\sigma_i(B) = \sigma_i(A)$.

Definition 1.6.8. Let A be a cell. Define $\bar{A} = \bigcup_{B \leq A} B$.

Proposition 1.6.9 ([AB08], Proposition 1.24.)**.** *$\bar{A}$ is the closure of A in V.*

Definition 1.6.10. The cells of maximal dimension are called *chambers*. A *panel* is a cell of codimension 1. If a panel A is a face of a chamber C, then its support hyperplane H is called a *wall* of C

Definition 1.6.11. Let the hyperplane arrangement $\mathcal{H}$ be W-invariant. We write $\Sigma(W, V) = \Sigma(\mathcal{H})$ for the set of cells. Then W acts as a group of poset automorphism on $\Sigma(W, V)$.

Proposition 1.6.12 ([AB08], Proposition 1.107.)**.** *The poset $\Sigma(W, V)$ is a simplicial complex.*

Proposition 1.6.13 ([AB08], Proposition 1.108.)**.** *The geometric realization $|\Sigma(W, V)|$ is canonically homeomorphic to a sphere of dimension $\operatorname{rank}(W, V) - 1$.*

1.7. Coxeter Groups

This section is based on [Bou02] Chapter IV and [AB08] Section 2.4.

Definition 1.7.1. Let W be a group with identity 1_W and S be a finite set of generators of W of order 2, such that $1_W \notin S$. The following condition on (W, S) is called the *Coxeter condition*:

$$W = \langle S \mid (st)^{m(s,t)} = 1 \rangle,$$

where $m(s, t)$ is the order of st and there is one relation for each pair s, t in S with $m(s, t) < \infty$.

Definition 1.7.2. A group W is a *Coxeter group*, if the Coxeter condition holds for W. We call the pair (W, S) a *Coxeter system.* The Matrix

$$M = (m(s,t))_{s,t \in S}$$

is called the *Coxeter matrix* of (W, S) and the cardinality $|S|$ will be called the *rank* of the Coxeter system (W, S).

Remark 1.7.3. The entries of a Coxeter matrix are given by

$$m(s,s) = 1 \text{ and } 2 \leq m(s,t) = m(t,s) \leq \infty \text{ for } s \neq t.$$

For $m(s,t) = \infty$ there is no relation between s and t.

Definition 1.7.4. Let (W, S) be a Coxeter system. For $w \in W$ the *length* $l_S(w) = l(w)$ (with respect to S) is the smallest integer $r \geq 0$, such that w can be written as a product of r elements from S. A *reduced decomposition* of w (with respect to S) is any sequence $(s_1, ..., s_r)$ of elements of S such that $w = s_1 \dots s_r$ and $l(w) = r$.

Remark 1.7.5. Therefore 1_W is the unique element of length 0 and the elements in S are those elements of length 1.

Definition 1.7.6. For any subset $J \subseteq S$, we set $W_J = \langle J \rangle$. We call W_J a *standard parabolic subgroup.*

Proposition 1.7.7 ([AB08], Proposition 2.14.)**.** *Let (W, S) be a Coxeter system and W_J be a standard parabolic subgroup for some subset $J \subseteq S$. For any $w \in W_J$ we have $l_J(w) = l_S(w)$.*

Lemma 1.7.8 ([AB08], Lemma 2.15.)**.** *Let (W, S) be a Coxeter system. Given $J \subseteq S$, $w \in W_J$, and $s \in S \setminus J$, we have $l(sw) = l(w) + 1$.*

Definition 1.7.9. A Coxeter system (W, S) of rank n is called *spherical* (of dimension $n - 1$) if W is finite.

1.7.1. Coxeter Diagram

The definitions are taken from [AB08] Section 1.5.6.

Definition 1.7.10. Let (W, S) be a Coxeter system with corresponding Coxeter matrix

$$M = (m(s,t))_{s,t \in S}.$$

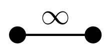
We define the *Coxeter diagram* to be a graph with $n = |S|$ vertices, so that every vertex corresponds to an element $s \in S$. The edges are given by the following rules:

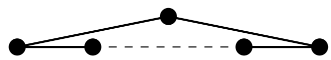
- We connect two distinct vertices s and t by an edge if and only if $m(s,t) \geq 3$. So the vertices s and t are not connected if $m(s,t) = 2$.
- If $m(s,t) \geq 4$, then we label the edge with the number $m(s,t)$.
- When we do not label an edge, then the number $m(s,t)$ equals 3.

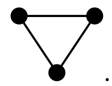
Definition 1.7.11. We call a Coxeter system *irreducible* if the corresponding Coxeter diagram is connected.

Example 1.7.12. We only work in the situation where we have a diagram of type A_n or $\tilde{A}_n$. The diagrams of type A_n and $\tilde{A}_n$:

A_n $(n \in \mathbb{N})$ 

$\tilde{A}_1$ 

$\tilde{A}_n$ $(n \geq 2)$ 

Actually we mainly work with A_2  and $\tilde{A}_2$  .

1.8. Coxeter Complexes

In this section we follow Section 3.1 in [AB08].

Let (W,S) be a Coxeter system, where S is finite.

Definition 1.8.1. A *standard coset* in W is a coset wW_J, where $w \in W$ and $W_J = \langle J \rangle$ for some subset $J \subseteq S$.

Definition 1.8.2. Let

$$\Sigma(W,S) = \{wW_J \mid w \in W \text{ and } J \subseteq S\}.$$

Then $\Sigma(W,S)$ is a poset, when we define the order by $B \leq A$ in $\Sigma(W,S)$ if and only if $B \supseteq A$ as subsets of W. We call $\Sigma(W,S)$ the *Coxeter complex* associated to (W,S).

Definition 1.8.3. The Coxeter complex $\Sigma(W,S)$ is called *spherical* if W is finite.

Theorem 1.8.1 ([AB08], Theorem 2.65.)**.** *A spherical Coxeter complex $\Sigma(W,S)$ is isomorphic to the simplicial complex $\Sigma(W,V)$, where W is viewed as finite reflection group of the Euclidean vector space $V = \mathbb{R}^S$ equipped with the symmetric bilinear form $B(e_s,e_t) = -\frac{\pi}{m(s,t)}$; here $M = (m(s,t))_{s,t \in S}$ is the associated Coxeter matrix and $(e_s)_{s \in S}$ denotes the standard basis of V.*

Remark 1.8.4. Due to 1.8.1 and 1.6.13 a spherical Coxeter complex is isomorphic to $\Sigma(W, V)$ whose geometric realization is homeomorphic to a sphere.

Definition 1.8.5. The Coxeter complex $\Sigma(W, S)$ has a *canonical type function* with values in S. This is the function τ defined by $\tau(wW_J) = S \setminus J$ for $w \in W$ and $J \subseteq S$. Equivalently, the simplex wW_J has cotype J.

Theorem 1.8.2 ([AB08], Theorem 3.5)**.** *The poset $\Sigma(W, S)$ is a simplicial complex. Moreover, it is a thin chamber complex of rank equal to $|S|$, it is colorable, and the action of W on $\Sigma(W, S)$ is type-preserving.*

Remark 1.8.6. The chambers in $\Sigma(W, S)$ are the singletons $\{w\}$ for elements $w \in W$. The panels are simplices of the form $w\langle s\rangle = \{w, ws\}$ for $w \in W$ and $s \in S$. So each panel is a face of exactly two chambers, for instance, $w\langle s\rangle = \{w, ws\}$ is a face of w and ws.

Definition 1.8.7. Let $\Sigma(W, S)$ be a Coxeter complex. We call the chamber $C := 1_W$ the *fundamental chamber*.

Definition 1.8.8. A simplicial complex Σ is called a *Coxeter complex* if it is isomorphic to $\Sigma(W, S)$ for some Coxeter system (W, S). It is called a *spherical Coxeter complex* if it is finite.

Proposition 1.8.9 ([AB08], Corollary 3.17)**.** *$\Sigma(W, S)$ is completely determined by its underlying chamber system. More precisely, the simplices of $\Sigma(W, S)$ are in $1-1$ correspondence with the residues in $\mathcal{C}(\Sigma(W, S))$, ordered by reverse inclusion. Here a simplex A corresponds to the residue $\mathcal{C}(\Sigma(W, S))_{\geq A}$, consisting of the chambers having A as a face.*

Definition 1.8.10. Let C and $-C$ be two adjacent chambers in a Coxeter complex Σ. A *root* of Σ is a subcomplex

$$\alpha = \{u \in \Sigma \mid \exists D \in \mathcal{C}(\Sigma) : u \subseteq D \text{ and } d(D, C) < d(D, -C)\},$$

where the chambers of α are given by the set

$$\mathcal{C}(\alpha) = \{D \in \mathcal{C}(\Sigma) \mid d(D, C) < d(D, -C)\}.$$

The opposite root

$$-\alpha = \{u \in \Sigma \mid \exists D \in \mathcal{C}(\Sigma) : u \subseteq D \text{ and } d(D, C) > d(D, -C)\}$$

to α is the subcomplex with set of chambers equal to

$$\mathcal{C}(-\alpha) = \{D \in \mathcal{C}(\Sigma) \mid d(D, C) > d(D, -C)\}.$$

The intersection $\partial\alpha := \alpha \cap -\alpha$ of two opposite roots will be called the *wall* bounding $\pm\alpha$.

Theorem 1.8.3 ([AB08], Theorem 3.65.)**.** *A thin chamber complex Σ is a Coxeter complex if and only if every pair of adjacent chambers is separated by a wall.*

1.9. Buildings

This section is based on Section 4.1 in [AB08].

Definition 1.9.1. A *building* is a simplicial complex Δ, that is a union of subcomplexes Σ (called apartments) satisfying the following axioms:

- (B0) Each apartment Σ is a Coxeter complex.
- (B1) For any two simplices $A, B \in \Delta$, there is an apartment Σ containing both of them.
- (B2) If Σ and Σ' are two apartments containing A and B, then there is an isomorphism $\Sigma \to \Sigma'$ fixing A and B pointwise.

Remark 1.9.2. In axiom $(B2)$ we can take the empty set for the simplices A and B. Then the axiom implies that any two apartments in the building are isomorphic. Therefore a building Δ is a finite-dimensional simplicial complex and its dimension is the common dimension of its apartments. Moreover Δ is a chamber complex.

Definition 1.9.3. Let Δ be a building. Any set $\mathcal{A}$ of subcomplexes Σ satisfying the axioms in 1.9.1 will be called a *system of apartments* for Δ.

Proposition 1.9.4 ([AB08], Proposition 4.7)**.** *All apartments in a building have the same Coxeter matrix.*

Definition 1.9.5. If Δ is a building where all apartments have the Coxeter matrix $M = (m(s,t))_{s,t\in S}$, then we call M the *Coxeter matrix* of Δ. Similarly, we speak of the *Coxeter diagram* of Δ. The *rank* of Δ is the cardinality of S.

Definition 1.9.6. Let (W, S) be a Coxeter system with Coxeter matrix M. A building Δ is of *type* (W, S) (or of type M) if Δ comes equipped with a type function having values in S such that the Coxeter matrix of Δ is M. We then say that W is the *Weyl group* of Δ.

Definition 1.9.7. A building is called *thick* if every panel is a face of at least three chambers.

Proposition 1.9.8 ([AB08], Corollary 4.11.)**.** *A building Δ is completely determined by its underlying chamber system. More precisely, the simplices of Δ are in $1-1$ correspondence with the residues in $\mathcal{C} := \mathcal{C}(\Delta)$, ordered by reverse inclusion. Here a simplex A corresponds to the residue $\mathcal{C}_{\geq A}$, consisting of the chambers having A as a face.*

Theorem 1.9.1 ([AB08], Theorem 4.54.)**.** *If Δ is a building, then the union of any family of apartment systems is again an apartment system. Consequently, Δ admits a largest system of apartments.*

Definition 1.9.9. The maximal apartment system will be called the *complete system of apartments.*

Definition 1.9.10. Let Δ be a building and Δ' a chamber subcomplex of Δ. Then Δ' is a *subbuilding* of Δ if Δ' is a building in its own right and every apartment in the complete system of apartments of Δ' is an apartment in the complete system of apartments of Δ.

Proposition 1.9.11 ([AB08], Proposition 4.63.)**.** *Let Δ be a building of type M. A chamber subcomplex Δ' of Δ is a subbuilding if and only if Δ' is a building in its own right and its Coxeter matrix is M.*

Definition 1.9.12. A building is called *spherical* if its apartments are spherical Coxeter complexes.

Theorem 1.9.2 ([AB08], Theorem 4.70.)**.** *A spherical building admits a unique system of apartments.*

1.9.1. The Weyl distance

The definition can be found in [AB08] Section 4.8.

We assume in this subsection that Δ is a building of type (W, S).

Proposition 1.9.13 ([AB08], Proposition 4.81.)**.** *There is a function $\delta : \mathcal{C}(\Delta) \times \mathcal{C}(\Delta) \to W$ with the following properties:*

a) Given a minimal gallery $\Gamma = (C_0, ..., C_d)$ of type $s(\Gamma) = (s_1, ..., s_d)$, $\delta(C_0, C_d)$ is the element $w = s_1...s_d$ represented by $s(\Gamma)$.

b) Let C and D be chambers, and let $w = \delta(C, D)$. The function $\Gamma \mapsto s(\Gamma)$ gives a $1-1$ correspondence between minimal galleries from C to D and reduced decompositions of w.

Definition 1.9.14. The function δ of 1.9.13 is called the *Weyl distance function* associated to Δ.

1.9.2. Group actions on buildings

Based on [AB08] Section 6.1.

Let Δ be a building of type (W, S) and G be a group acting type-preservingly on Δ. Suppose we have a system of apartments $\mathcal{A}$ which is G-invariant, in particular, if Σ is an apartment in $\mathcal{A}$, then its image $g\Sigma$ is again in $\mathcal{A}$ for all elements $g \in G$.

Definition 1.9.15. The G-action is *strongly transitive* (with respect to $\mathcal{A}$) if G acts transitively on the set of pairs (Σ, C) consisting of an apartment $\Sigma \in \mathcal{A}$ and a chamber $C \in \Sigma$.

Remark 1.9.16. The action of G is strongly transitive if and only if the action is transitive on the set of chambers and the stabilizer of a given chamber C acts transitive on the set of apartments in $\mathcal{A}$ containing C.

Definition 1.9.17. Assume we have a strongly transitive G-action on Δ (with respect to $\mathcal{A}$), and choose an arbitrary pair (Σ, C) of an apartment $\Sigma \in \mathcal{A}$ and a chamber $C \in \Sigma$. We call C the *fundamental chamber* and Σ the *fundamental apartment*.

Definition 1.9.18. The action of G on Δ is *Weyl transitive* if for each $w \in W$, the action is transitive on the set of ordered pairs (C, D) of chambers with Weyl distance $\delta(C, D) = w$.

1.9.3. The building $\Delta(G, B)$

The definitions are taken from [AB08] Section 6.2.

Definition 1.9.19. Let G be a group, B a subgroup of G, (W, S) a Coxeter system and $C : W \to B \setminus G/B$ a bijection which satisfies the following condition:

(B) For all $s \in S$ and $w \in W$ it is $C(sw) \subseteq C(s)C(w) \subseteq C(sw) \cup C(w)$.

If $l(sw) = l(w) + 1$, then $C(s)C(w) = C(sw)$. Then the bijection C is said to provide a *Bruhat decomposition* of type (W, S) for (G, B).

Definition 1.9.20. If we have a Bruhat decomposition of type (W, S) for (G, B) and $J \subseteq S$, then the *standard parabolic subgroup* P_J is defined to be the union $P_J := \bigcup_{w \in W_J} C(w)$.

Definition 1.9.21. Given a Bruhat decomposition for (G, B) we define

$$\Delta(G, B) = \{gP_J \mid g \in G \text{ and } P_J \text{ a standard parabolic subgroup } \}.$$

This defines a poset when we define the order by reverse inclusion.

Proposition 1.9.22 ([AB08], Proposition 6.34.)**.** *Given a Bruhat decomposition for (G, B), the poset $\Delta(G, B)$ is a building, and the natural action of G on $\Delta(G, B)$ by left translation is Weyl transitive and has B as the stabilizer of a fundamental chamber. Conversely, if a group G admits a Weyl transitive action on a building Δ and B is the stabilizer of a fundamental chamber, then (G, B) admits a Bruhat decomposition and Δ is canonically isomorphic to $\Delta(G, B)$.*

Definition 1.9.23. A Bruhat decomposition for (G, B) is called *thick* if the building $\Delta(G, B)$ is thick.

Theorem 1.9.3 ([AB08], Theorem 6.43.)**.** *Suppose (G, B) admits a thick Bruhat decomposition.*

a) The standard parabolic subgroups are precisely the subgroups of G containing B.

b) If P is a standard parabolic subgroup and $gBg^{-1} \leq P$ for some $g \in G$, then $g \in P$.

c) Every standard parabolic subgroup is equal to its own normalizer, and no two of them are conjugate.

1.9.4. BN-Pairs

This subsection is based on Section 6.2.6 in [AB08].

Definition 1.9.24. Let G be a group. We call a pair of subgroups B and N of G a *BN-pair* if B and N generate G, the intersection $T := B \cap N$ is normal in N, and the quotient $W := N/T$ admits a set of generators S such that the following conditions hold:

- (BN1) For $s \in S$ and $w \in W$, $sBw \subseteq BswB \cup BwB$.
- (BN2) For $s \in S$, $sBs^{-1} \nleq B$.

The group W is called the *Weyl group* associated to the BN-pair.

Theorem 1.9.4 ([AB08], Theorem 6.56.). *a) Given a BN-pair in G, the generating set S is uniquely determined, and (W, S) is a Coxeter system. There is a thick building $\Delta(G, B)$ that admits a strongly transitive G-action such that B is the stabilizer of a fundamental chamber and N stabilizes a fundamental apartment and is transitive on its chambers.*

b) Conversely, suppose a group G acts strongly transitively on a thick building Δ with fundamental apartment Σ and fundamental chamber C. Let B be the stabilizer of C, and let N be a subgroup of G that stabilizes Σ and is transitive on the chambers of Σ. Then (B, N) is a BN-pair in G, and Δ is canonically isomorphic to $\Delta(G, B)$.

Remark 1.9.25. The building $\Delta(G, B)$ in 1.9.4 is the building we defined in 1.9.21. The group G acts on $\Delta(G, B)$ by left translation. The fundamental apartment is

$$\Sigma = \{wP_J \mid w \in W \text{ and } P_J \text{ is a standard parabolic subgroup } \}.$$

1.9.5. The spherical building for SL_n

This subsection summarizes some results of Section 6.5 in [AB08].

We consider the special linear group $G = \mathrm{SL}_n(k)$, $n \in \mathbb{N}_{>1}$, over an arbitrary field k. Let $B \leq G$ be the upper-triangular group, i.e. the stabilizer of the standard flag

$$\langle e_1 \rangle < \langle e_1, e_2 \rangle < ... < \langle e_1, ..., e_n \rangle,$$

where $e_1, ..., e_n$ denotes the standard basis of k^n. Furthermore we define N to be the monomial subgroup of G, i.e. the stabilizer of the set of lines $\{\langle e_1 \rangle, ..., \langle e_n \rangle\}$. Then $T = B \cap N$ is the diagonal subgroup of G and $W = N/T$ can be identified with the symmetric group on n letters.

Proposition 1.9.26 ([AB08], Section 6.5.)**.** *Let k be a field, $n \in \mathbb{N}_{>1}$, denote by B be the upper-triangular group of $G = \mathrm{SL}_n(k)$ and let N be the monomial subgroup of G. Then (B, N) is a BN-pair for G. The associated building $\Delta(G, B)$ is isomorphic to the complex of flags of proper nonzero subspaces of k^n.*

1.10. Affine buildings

Definitions are taken from [AB08], [Wei09] and [MPW15].

1.10.1. Euclidean Reflection Groups

In this subsection we follow Section 10.1 in [AB08].

Let V be a Euclidean vector space, i.e. real vector space equipped with a positive definite inner product, of finite dimension $n \geq 1$.

Definition 1.10.1. An *affine subspace* of V is a coset of a linear subspace defined by a linear equation of the form $f = c$, where $f : V \to \mathbb{R}$ is a nonzero linear map and c is a constant. The *dimension* of this affine subspace is by definition the dimension of the corresponding linear subspace. If the corresponding linear subspace is a hyperplane, then we call the affine subspace an *affine hyperplane*.

Definition 1.10.2. For a vector $v \in V$ denote by τ_v the *translation* $\tau_v : V \to V, w \mapsto w + v$. The *group of affine automorphisms* of V is the semidirect product $\mathrm{Aff}(V) = V \rtimes \mathrm{GL}(V)$, where we identify V with the normal subgroup consisting of all translations. Furthermore the *group of affine isometries* of V is given by $V \rtimes O(V)$, where $O(V) \leq \mathrm{GL}(V)$ denotes the orthogonal group.

Definition 1.10.3. Let $H = x + H_0$ be an affine hyperplane, with $x \in V$ and H_0 is a linear hyperplane in V. Further, let s_{H_0} be the orthogonal reflection corresponding to the hyperplane H_0. Then we call $s_H := \tau_x s_{H_0} \tau_{-x}$ the *reflection* with respect to H.

Remark 1.10.4. The reflection s_H depends only on the hyperplane H.

Definition 1.10.5. A group W of affine isometries of V is an *affine reflection group* if there is a set $\mathcal{H}$ of affine hyperplanes in V satisfying:

1. W is generated by the reflections s_H for $H \in \mathcal{H}$.

2. $\mathcal{H}$ is W-invariant.

3. $\mathcal{H}$ is locally finite, in the sense that every point of V has a neighborhood that meets only finitely many $H \in \mathcal{H}$.

Definition 1.10.6. For an affine reflection group W we define similar to the finite reflection group case the following: *cells* are nonempty sets defined by linear equalities or strict inequalities, one for each $H \in \mathcal{H}$. This means if H is defined by $f = c$, then the definition of A involve either the same equality or one of the inequalities $f > c$ or $f < c$. The *support* of a cell is the intersection of all affine hyperplanes that corresponds to a linear equality in the definition of the cell. The *dimension* of a cell is the dimension of its support. The cells of maximal dimension n are called *chambers* and those of dimension $n - 1$ are called *panels*. We have a face relation on the set of cells. The supports of the panels of a chamber C are called the *walls* of C.

Proposition 1.10.7 ([AB08], section 10.1.3.)**.** *Let W be an affine reflection group, choose a chamber C and let S be the set of reflections with respect to the walls of C. Then the following holds:*

a) W is simply transitive on the chambers.

b) W is generated by S.

c) $\mathcal{H}$ necessarily consists of all affine hyperplanes H with $s_H \in W$.

d) (W, S) is a Coxeter system.

Theorem 1.10.1 ([AB08], Theorem 10.8.)**.** *With the notation from the previous proposition 1.10.7.*

a) C has only finitely many walls, and hence S is finite.

b) There are only finitely many linear hyperplanes H_0 such that $\mathcal{H}$ contains an affine hyperplane $H = x + H_0$ for some $x \in V$.

c) Let $\bar{W} \leq \mathrm{GL}(V)$ be the set of linear parts of the elements in W, i.e. the image of W under the projection $\mathit{Aff}(V) \twoheadrightarrow \mathrm{GL}(V)$. Then $\bar{W}$ is a finite reflection group.

Definition 1.10.8. We call the affine reflection group W *essential* if the associated finite reflection group $\bar{W}$ is essential.

Definition 1.10.9. The affine reflection group W will be called *irreducible* if the Coxeter diagram of (W, S) is connected.

Definition 1.10.10. A *Euclidean reflection group* is an essential, irreducible, infinite, affine reflection group.

Let W denote a Euclidean reflection group.

Definition 1.10.11. We denote by $\Sigma(W, V)$ the poset of cells together with an element which is defined to be a face of each cell.

Remark 1.10.12. The poset of cells has no smallest element and hence it can not be a simplicial complex. The poset $\Sigma(W, V)$ is a simplicial complex and its geometric realization is homeomorphic to V.

Proposition 1.10.13 ([AB08], Proposition 10.13.)**.**

a) The simplicial complex $\Sigma(W, V)$ is isomorphic to the Coxeter complex $\Sigma(W, S)$ of the Coxeter system (W, S) in 1.10.7.

b) $\Sigma(W, V)$ triangulates V.

1.10.2. Euclidean Coxeter complexes and affine buildings

The following is based on Section 10.2 and Chapter 11 in [AB08].

Definition 1.10.14. A Coxeter complex Σ will be called *Euclidean* if it is isomorphic to $\Sigma(W, V)$ for some Euclidean reflection group (W, V).

Lemma 1.10.15 ([AB08], Lemma 10.36.)**.** *Let (W, V) and (W', V') be Euclidean reflection groups. Let $\phi : \Sigma(W, V) \to \Sigma(W', V')$ be a simplicial isomorphism. Then the composite bijection*

$$V \cong |\Sigma(W, V)| \xrightarrow{|\phi|} |\Sigma(W', V')| \cong V'$$

is an affine isomorphism whose linear part g satisfies $\langle gv, g\bar{v}\rangle = \lambda\langle v, \bar{v}\rangle$ for some positive constant λ and all $v, \bar{v} \in V$.

Remark 1.10.16. By the previous Lemma 1.10.15 we have a well defined equivalence class of metrics on the geometric realization of an Euclidean Coxeter complex Σ, in particular two metrics are equivalent if one is a positive scalar multiple of the other. We choose a canonical representative of this equivalence class to view $|\Sigma|$ as metric space. Then the abstract isomorphism $\phi : \Sigma \to \Sigma'$ induces an isometry $|\Sigma| \to |\Sigma'|$.

Definition 1.10.17. An *affine building* is a building whose apartments are Euclidean Coxeter complexes.

Remark 1.10.18. Affine buildings are also called *Euclidean buildings*, because one can view the geometric realization of an apartment in an affine building as some Euclidean space $\mathbb{R}^d$ (see 1.10.15).

1.10.3. The spherical building at infinity

This section summarizes some sections in Chapter 11 in [AB08] where the construction of the building at infinity is described. For more details on the construction read the relevant sections in, for example, [AB08], [Gar97] or with another description for buildings in [Wei09] and [Ron09].

We start with a Euclidean building Δ equipped with its complete system of apartments $\mathcal{A}$. If the apartment system is not complete the construction using only these apartments yields a subcomplex of the building at infinity, which is not always a building. Later in this section we will state a condition which ensures that we get a building although we use a non-complete system of apartments.

In this section we call the geometric realization X of the Euclidean building Δ itself a Euclidean building and the subset $E = |\Sigma| \subseteq X$ for $\Sigma \in \mathcal{A}$ an apartment of X.

Definition 1.10.19. We can use the Euclidean structures of the apartments of X to define a *distance* in X by

$$d(x, y) := d_E(x, y)$$

for two points $x, y \in X$ and an apartment E containing both.

Remark 1.10.20. If we have another apartment E' containing x and y, then we can find an isometry $E \to E'$ which fixes both points x and y. Hence the distance is independent of the choice of an apartment. We obtain a metric $d : X \times X \to \mathbb{R}$ (see [AB08], Theorem 11.16.).

Definition 1.10.21. Choose an identification of an apartment Σ with the complex $\Sigma(W, V)$ associated to a Euclidean reflection group. This gives an identification of $E = |\Sigma|$ with V. Let $\bar{W}$ be the finite reflection group consisting of the linear parts of the elements of W. The *conical cells* based at a point $x \in E$ are the translates $\mathfrak{U} = x + \mathfrak{D}$, where $\mathfrak{D}$ is a cell associated to $\bar{W}$. We will call $\mathfrak{D}$ the *direction* of $\mathfrak{U}$. If the $\bar{W}$-cell $\mathfrak{D}$ is a chamber, then the conical cell $x + \mathfrak{D}$ will be called a *sector*. If $\mathfrak{C}$ and $\mathfrak{C}'$ are sectors with $\mathfrak{C}' \subseteq \mathfrak{C}$, then $\mathfrak{C}'$ is called a *subsector* of $\mathfrak{C}$.

Definition 1.10.22. A *conical cell* in X is a subset $\mathfrak{U}$ that is contained in some apartment E and is a conical cell in E. Similarly, a *sector* in X is a subset $\mathfrak{C}$ that is contained in some apartment E and is a sector in E.

Proposition 1.10.23 ([AB08], Proposition 11.62.)**.** *If $\mathfrak{U}$ is a conical cell in some apartment E, then $\mathfrak{U}$ is a conical cell in every apartment E' that contains it.*

Definition 1.10.24. A *ray* in X is a subset $\mathfrak{r}$ that is isometric to the half-line $[0, \infty)$. The point $x \in \mathfrak{r}$ that corresponds to 0 under the unique such isometry will be called the *basepoint* of $\mathfrak{r}$.

Remark 1.10.25. A ray is convex and hence it is contained in some apartment of X (see [AB08], Theorem 11.53.). As subset of the apartment the ray is of the form

$$\{(1-t)x + ty \mid t \geq 0\}$$

for some $x \neq y$.

Definition 1.10.26. We call two rays $\mathfrak{r}, \mathfrak{s}$ *parallel* if there exists a number M such that for each $y \in \mathfrak{r}$ there is a $z \in \mathfrak{s}$ with $d(y,z) < M$ and similarly with the roles of $\mathfrak{r}$ and $\mathfrak{s}$ reversed.

Remark 1.10.27. If we have two rays $\mathfrak{r}, \mathfrak{s}$ in some apartment E, then they are parallel if and only if there is a translation of E taking one to the other. The relation of parallelism is an equivalence relation.

Definition 1.10.28. An equivalence class of rays will be called an *ideal point* of X. Let X_∞ denote the set of ideal points.

Lemma 1.10.29 ([AB08], Lemma 11.72.)**.** *Given a point x and a ray $\mathfrak{s}$, there is a unique ray $\mathfrak{r}$ that is based at x and parallel to $\mathfrak{s}$.*

Definition 1.10.30. If $\mathfrak{r}$ is a ray with basepoint x and representing the ideal point e we write $[x,e)$ for the ray $\mathfrak{r}$. The *open ray* (x,e) is defined to be the set $[x,e) \setminus \{x\}$.

Definition 1.10.31. Let $\mathfrak{U}$ be a conical cell in X based at some point $x \in X$. The *face* of $\mathfrak{U}$ at infinity is the set

$$\mathfrak{U}_\infty = \{e \in X_\infty \mid (x,e) \subseteq \mathfrak{U}\}.$$

Definition 1.10.32. Let F be a face of a conical cell at infinity and x its cone point. The *open join* is defined as follows:

$$x \star F := \begin{cases} \{x\} & \text{if } F = \emptyset, \\ \bigcup\limits_{e \in F} (x,e) & \text{otherwise} \end{cases} .$$

Remark 1.10.33. We can recover a conical cell $\mathfrak{U}$ from its cone point x and its face at infinity $F = \mathfrak{U}_\infty$. In particular $\mathfrak{U} = x \star F$.

Definition 1.10.34. An *ideal simplex* of X is a subset F of X_∞ such that $F = \mathfrak{U}_\infty$ for some conical cell $\mathfrak{U}$.

Lemma 1.10.35 ([AB08], Lemma 11.75.)**.** *If F is an ideal simplex and x is an arbitrary point of X, then there is a conical cell $\mathfrak{U}$ based at x such that $F = \mathfrak{U}_\infty$. Consequently, there is a $1-1$ correspondence between the set of ideal simplices of X and the set of conical cells based at any given point $x \in X$.*

Lemma 1.10.36 ([AB08], Lemma 11.77.)**.** *Two sectors of X have the same face at infinity if and only if they have a common subsector.*

Definition 1.10.37. Let Δ_∞ be the set of ideal simplices of X.

Remark 1.10.38. In order to define a face relation on Δ_∞ we want to use the $1-1$ correspondence in 1.10.35 and a face relation on the set of conical cells in X, that is defined as follows:

We already have a face relation on the set of conical cells in an apartment E based at a given point x. We extend this face relation to conical cells in X based at x by saying that $\mathfrak{U}'$ is a face of $\mathfrak{U}$ if $\mathfrak{U}'$ is contained in the closure of $\mathfrak{U}$ and is a face of $\mathfrak{U}$ in some apartment containing $\mathfrak{U}$. In this case $\mathfrak{U}'$ is a face of $\mathfrak{U}$ in every apartment containing $\mathfrak{U}$.

Definition 1.10.39. Given ideal simplices F and F', we call F' a *face* of F if $x \star F'$ is a face of $x \star F$ for some $x \in X$.

Remark 1.10.40. If F' is a face of F, then $x \star F'$ is a face of $x \star F$ for every $x \in X$.

Definition 1.10.41. Let $E = |\Sigma|$ be an apartment of X. We define

$$\Sigma_\infty = \{F \in \Delta_\infty \mid F = \mathfrak{U}_\infty \text{ for some conical cell } \mathfrak{U} \text{ in } E\}$$

and call it an *apartment* of Δ_∞.

Theorem 1.10.2 ([AB08], Theorem 11.79.)**.** *The poset Δ_∞ is a spherical building. Its apartments are in $1-1$ correspondence with those of X.*

Remark 1.10.42. From the proof of Theorem 11.79 given in [AB08] it follows that Σ_∞ is a subcomplex of Δ_∞ and it is a finite Coxeter complex. In particular, if we identify Σ with $\Sigma(W, V)$ for some Euclidean reflection group (W, V), then Σ_∞ is isomorphic to $\Sigma(\bar{W}, V)$, where $\bar{W}$ is the associated finite reflection group.

Definition 1.10.43. Δ_∞ is called the *building at infinity* associated to Δ.

Now we assume the apartment system $\mathcal{A}$ to be incomplete and we want to do the same construction as we did for the building at infinity but using only the apartments in $\mathcal{A}$.

Definition 1.10.44. Let $\mathcal{A}_\infty$ be the set of apartments Σ_∞ in Δ_∞ with $\Sigma \in \mathcal{A}$. Further, define

$$\Delta_\infty(\mathcal{A}) = \bigcup_{\Sigma \in \mathcal{A}} \Sigma_\infty.$$

Remark 1.10.45. $\Delta_\infty(\mathcal{A})$ is a subcomplex of Δ_∞ and we want to know whether it is a building with apartment system $\mathcal{A}_\infty$.

Definition 1.10.46. We call a sector in X an *$\mathcal{A}$-sector* if it is contained in an apartment in $\mathcal{A}$.

Proposition 1.10.47 ([AB08], Proposition 11.89.). *$\Delta_\infty(\mathcal{A})$ is a building with $\mathcal{A}_\infty$ as system of apartments if and only if $\mathcal{A}$ has the following property: For any two $\mathcal{A}$-sectors, there is an apartment in $\mathcal{A}$ containing a subsector of each of them.*

Definition 1.10.48. We call an apartment system $\mathcal{A}$ *good* if it satisfies the condition in the previous Proposition 1.10.47.

Proposition 1.10.49 ([AB08], Corollary 11.92.). *There is a $1-1$ correspondence between subbuildings of Δ_∞ and pairs $(X', \mathcal{A})$, with X' is a subbuilding of X and $\mathcal{A}$ a good apartment system for X'.*

Proposition 1.10.50 ([AB08], Proposition 11.93.). *Let $\mathcal{A}$ be a good system of apartments for a Euclidean building X. If X is thick, then the building at infinity $\Delta_\infty(\mathcal{A})$ is thick.*

2. Bruhat-Tits Buildings

In [Wei09] and [MPW15] a Bruhat-Tits building is defined to be an affine building whose "building at infinity" satisfies the Moufang property. See 2.2.3 for the exact definition. The building associated to the projective general linear group over a field with discrete valuation is an example of a Bruhat-Tits building. It is an affine building of type $\tilde{A}_{n-1}$, where n is the dimension of the corresponding vector space over the field with discrete valuation. This follows from Section 6.9.3 in the book [AB08]. Moreover, it is shown there, that the building at infinity is the building associated to the n-dimensional vector space, where we forget that we have a discrete valuation on the field. Therefore the building at infinity is a spherical building of type A_{n-1}. From Section 7.3.4 in [AB08] we know that this building satisfies the Moufang property.

2.1. Discrete valuations

The definitions in this section can be found in various books. For example in [AB08] Section 6.9.1, [Ser80] Chapter 1, Section 1.6 and [Ron09] Chapter 9, Section 2. The part about the function field case is basically taken from [NX09] Section 1.5.

Definition 2.1.1. Let K be a field. A *discrete valuation* on K is a surjective homomorphism $\nu : K^\times \to \mathbb{Z}$, together with the convention that $\nu(0) = \infty$, such that

$$\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$$

for all $a, b \in K$. Sometimes we write K_ν for a field K with discrete valuation ν.

Definition 2.1.2. Let $\pi \in K^\times$ be a *uniformizer*, i.e. an element with $\nu(\pi) = 1$.

Definition 2.1.3. We denote by $\mathcal{O}_\nu := \{a \in K \mid \nu(a) \geq 0\}$ the *valuation ring* of K with respect to ν.

Remark 2.1.4. The invertible elements in the valuation ring are given by $\mathcal{O}_\nu^\times = \{a \in K \mid \nu(a) = 0\}$.

Remark 2.1.5. Let K be a field with discrete valuation and π a uniformizer. Then every element $a \in K^\times$ can be written uniquely as $a = u\pi^z$ for some $u \in \mathcal{O}_\nu^\times$ and $z \in \mathbb{Z}$.

Lemma 2.1.6. *The valuation ring $\mathcal{O}_\nu$ has a unique maximal ideal $\pi\mathcal{O}_\nu$.*

Proof. For each element $a \in K^\times$ we have

$$a\mathcal{O}_\nu = \mathcal{O}_\nu a = \pi^{\nu(a)}\mathcal{O}_\nu = \{x \in K \mid \nu(x) \geq \nu(a)\}.$$

Hence the non-trivial ideals of $\mathcal{O}_\nu$ are of the form $\pi^r\mathcal{O}_\nu$ for some $r \in \mathbb{N}$. $\square$

Definition 2.1.7. The unique maximal ideal $p = \pi\mathcal{O}_\nu$ is called a *place* of K. The field $k_p := \mathcal{O}_\nu/\pi\mathcal{O}_\nu$ is called the *residue field.*

Remark 2.1.8. If we have a place p, then it corresponds to a valuation of K. We denote this valuation by ν_p.

Definition 2.1.9. Let S be a non-empty finite set of places of K. Then

$$\mathcal{O}_S = \{a \in K \mid \nu_p(a) \geq 0 \text{ for all places } p \notin S\} = \bigcap_{p \notin S} \mathcal{O}_{\nu_p}$$

is the set of elements in K having only poles in S. It is called the ring of *S-integers.*

2.1.1. Completion

Here we follow Section 6.9.1 in [AB08].

Given a field K with a discrete valuation ν we can consider the map

$$K \to \mathbb{R},\ a \mapsto |a| := e^{-\nu(a)}.$$

Definition 2.1.10. For an element $a \in K$ the real value $|a| := e^{-\nu(a)}$ is called the *absolute value* of a.

Remark 2.1.11. The absolute value satisfies $|ab| = |a||b|$ and $|a + b| \leq max\{|a|, |b|\}$ for all $a, b \in K$.

Definition 2.1.12. We define a *metric on* K by $d(a, b) := |a - b|$ for $a, b \in K$.

Definition 2.1.13. The field K is called *complete*, if every Cauchy sequence converges to a limit in K. If K is not complete, then we can form the *completion* $\hat{K}$ of K by adjoining all limits of Cauchy sequences to K.

Remark 2.1.14. The field operations and the discrete valuation ν extend to $\hat{K}$ by continuity, hence $\hat{K}$ is again a field with discrete valuation. The valuation ring of $\hat{K}$ is the completion $\hat{\mathcal{O}}_\nu$ of $\mathcal{O}_\nu$ and the residue field is the same field k_p as that for $\mathcal{O}_\nu$.

Example 2.1.15. Let $\mathbb{F}_q$ be a finite field with q elements. The completion of the rational function field $\mathbb{F}_q(t)$ with valuation corresponding to the irreducible polynomial $t \in k[t]$ can be identified with the field of formal Laurent series $\mathbb{F}_q((t))$. Similarly, the completion with respect to the valuation ν_∞ is $\mathbb{F}_q((\frac{1}{t}))$.

2.1.2. Discrete valuations on global function fields

The following is based on Section 1.5 in [NX09].

Definition 2.1.16. Let k be a field. A *function field* over k is a field extension K of k with at least one element $t \in K$ that is transcendental over k. Then we call k the *constant field* of K. If the constant field k is algebraically closed in K, i.e. $k = \{a \in K \mid a \text{ is algebraic over } k\}$, then k is called the *full constant field* of K. In the case that K is a finite extension of the rational function field $k(t)$, where $t \in K$ is a transcendental element over k, we say that K/k is an *algebraic function field of one variable* over k. If in addition the full constant field k is a finite field, then K/k is called a *global function field*.

From now on we denote by k the finite field $\mathbb{F}_q$ with q elements and by K/k a global function field.

Remark 2.1.17. When we consider a discrete valuation ν of K, then we have $\nu(a) = 0$ for all $a \in k^\times$, because for all $a \in k^\times$ we have $a^{q-1} = 1$ and hence $0 = \nu(1) = \nu(a^{q-1}) = (q-1)\nu(a)$, which is only possible for $\nu(a) = 0$.

Example 2.1.18. (see [NX09], Example 1.5.5.) Let $k = \mathbb{F}_q$ and $K = k(t)$ is the rational function field over k. The full constant field of $k(t)$ is k. Let $f \in k[t]$ be a monic, irreducible polynomial and define a discrete valuation of $k(t)$ corresponding to f as follows:

- for $a \in k[t] \setminus \{0\}$ with $f^m | a$ and $f^{m+1} \nmid a$ put $\nu_f(a) = m$;
- for $\frac{a}{b} \in k(t) \setminus \{0\}$ put $\nu_f(\frac{a}{b}) = \nu_f(a) - \nu_f(b)$;
- put $\nu_f(0) = \infty$.

Moreover, we define the discrete valuation ν_∞ of $k(t)$ by $\nu_\infty(\frac{a}{b}) = deg(b) - deg(a)$ for nonzero polynomials $a, b \in k[t]$ and $\nu_\infty(0) = \infty$.

Theorem 2.1.1 ([NX09], Theorem 1.5.8.)**.** *Every discrete valuation of the rational function field $k(t)$ is given by one of the discrete valuations in Example 2.1.18.*

Remark 2.1.19. By Theorem 2.1.1 and Example 2.1.18 we have a $1-1$ correspondence between places of $k(t)$ and $\{f \in k[x] \mid f \text{ is monic and irreducible }\} \cup \{\infty\}$.

Theorem 2.1.2 ([NX09], Theorem 1.5.13.)**.** *The residue field of every place of K/k is a finite extension (of an isomorphic copy) of k.*

Definition 2.1.20. We define the *degree* of a place p of K/k to be the degree of the residue field k_p over k.

Remark 2.1.21. When we consider a place p of the rational function field $K = k(t)$ associated to a monic, irreducible polynomial $f \in k[t]$, then the residue field k_p is isomorphic to $k[t]/(f)$. So the degree of the place p equals the degree of the associated polynomial f. Furthermore, the residue field of the place corresponding to ∞ is isomorphic to k and hence of degree 1. (see [NX09], Example 1.5.11.)

2.2. The definition of a Bruhat-Tits Building

The following definitions can be found in [MPW15], Chapter 1.

Definition 2.2.1. For any root α of a building Δ the corresponding *root group* U_α is the subgroup of $\mathrm{Aut}(\Delta)$ consisting of all elements that act trivially on each panel containing two chambers of α.

Definition 2.2.2. A building Δ is *Moufang* (or satisfies the Moufang condition) if

1. it is thick, spherical and irreducible,
2. its rank is at least 2,
3. for every root α the corresponding root group U_α acts transitively on the set of all apartments containing α.

Definition 2.2.3. A *Bruhat-Tits building* is a thick, irreducible, affine building whose building at infinity is Moufang.

2.3. The affine building for SL_n

In this section we follow [AB08], Chapter 6.9.

Let K denote a field with a discrete valuation ν and $n \in \mathbb{N}_{>1}$. Moreover, let $\mathcal{O}_\nu$ be the valuation ring, π a uniformizer and $k_p = \mathcal{O}_\nu/\pi\mathcal{O}_\nu$ be the residue field. We want to construct a BN-pair for $\mathrm{SL}_n(K)$ by using the BN-pair $(\bar{B}, \bar{N})$ we constructed in Section 1.9.5 for $\mathrm{SL}_n(k_p)$. Therefore we use the following diagram of matrix groups:

$$\begin{tikzcd} \mathrm{SL}_n(\mathcal{O}_\nu) \arrow[r, "\iota"] \arrow[d, "p"'] & \mathrm{SL}_n(K) \\ \mathrm{SL}_n(k_p) & \end{tikzcd}$$

In particular we define $B = \iota \circ p^{-1}(\bar{B})$, where $\bar{B}$ is the upper-triangular subgroup of $\mathrm{SL}_n(k_p)$, i.e.

$$B = \left\{ \begin{pmatrix} b_{1,1} & \dots & b_{1,n} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \dots & b_{n,n} \end{pmatrix} \in \mathrm{SL}_n(K) \mid \begin{array}{c} \nu(b_{i,i}) = 0 \text{ for } 1 \le i \le n, \\ \nu(b_{i,j}) \ge 0 \text{ for } 1 \le i < j \le n \\ \text{and } \nu(b_{i,j}) \ge 1 \text{ for } 1 \le j < i \le n \end{array} \right\}$$

and let N be the subgroup of $\mathrm{SL}_n(K)$ defined by monomial matrices. The intersection $T = B \cap N$ is the diagonal subgroup of $\mathrm{SL}_n(\mathcal{O}_\nu)$ and the Weyl group $W = N/T$ can be identified with the semidirect product $\mathbb{Z}^{n-1} \rtimes \bar{W}$, where $\bar{W} = \bar{B}/\bar{N}$ is the Weyl group corresponding to the spherical building for $\mathrm{SL}_n(k_p)$ (see Section 6.9.2 and Section 6.9.3 in [AB08]).

Proposition 2.3.1 ([AB08], Section 6.9.2.)**.** *The above defined subgroups B and N of $\mathrm{SL}_n(K)$ are a BN-pair for $\mathrm{SL}_n(K)$.*

Definition 2.3.2. Let (B, N) the above defined BN-pair for $G = \mathrm{SL}_n(K)$. The associated building $\Delta(G, B)$ will be called the *Bruhat-Tits building for* $\mathrm{SL}_n(K)$.

Proposition 2.3.3 ([AB08], Proposition 11.105.)**.** *Let $\Delta(G, B)$ be the Bruhat-Tits building for $\mathrm{SL}_n(K)$ defined in 2.3.2 and $X = |\Delta(G, B)|$.*

a) There is a sector $\mathfrak{C}$ in the fundamental apartment $E = |\Sigma|$ such that the stabilizer $\mathfrak{B}$ of $\mathfrak{C}_\infty$ is the upper-triangular subgroup of G.

b) The apartment system $\mathcal{A}$ associated to (G, B, N) is good. The subcomplex $\Delta_\infty(\mathcal{A})$ of Δ_∞ is therefore isomorphic to the spherical building associated to $\mathrm{SL}_n(K)$ in 1.9.5.

c) $\mathcal{A}$ is the complete apartment system if and only if K is complete with respect to the given valuation.

Lemma 2.3.4 ([AB08], Section 7.3.4.)**.** *The spherical building associated to $\mathrm{SL}_n(K)$ in 1.9.5 satisfies the Moufang property.*

Corollary 2.3.5. *The affine building associated to $\mathrm{SL}_n(K)$ in 2.3.2 is a Bruhat-Tits building.*

Proof. Follows from 2.3.3 and 2.3.4. □

2.4. Lattices

The following section is basically taken from Chapter 2, Section 1 in [Ser80] and Chapter 9, Section 2 in [Ron09].

In this section we assume K to be a field with discrete valuation ν and V is a n-dimensional vector space over K for some $n \in \mathbb{N}_{>1}$.

Definition 2.4.1. A *lattice* L of V (corresponding to the valuation ν) is a finitely generated $\mathcal{O}_\nu$-submodule of V which generates the K-vector space V.

Remark 2.4.2. The lattices of V correspond to bases of V, in particular, if $(b_1, ..., b_n)$ is a basis of V, then the corresponding lattice is $L = \mathcal{O}_\nu b_1 \oplus ... \oplus \mathcal{O}_\nu b_n = \langle b_1, ..., b_n \rangle_{\mathcal{O}_\nu}$.

Definition 2.4.3. Let $(e_1, ..., e_n)$ denote the standard basis of V. We call the lattice $L_s = \mathcal{O}_\nu e_1 \oplus ... \oplus \mathcal{O}_\nu e_n$ corresponding to the standard basis the *standard lattice.*

Remark 2.4.4. If we have a lattice L and some element $a \in K^\times$, then aL is again a lattice (since $a\mathcal{O}_\nu = \mathcal{O}_\nu a$).

Definition 2.4.5. We define an equivalence relation on the set of lattices by saying two lattices L and L' are *equivalent* if there exist an element $a \in K^\times$ such that $L = aL'$. By $[L]$ we denote the *equivalence class* of the lattice L.

2.5. The Bruhat-Tits Building for PGL_n

Based on [Ser80] Chapter 2, Section 1 and [Ron09] Chapter 9.

Let K be a field with discrete valuation ν and V be a n-dimensional vector space over K for some $n \in \mathbb{N}_{>1}$.

Definition 2.5.1. Let

$$P_\nu = \{[L] \mid L \text{ is a lattice of } V\}$$

the set of equivalence classes of lattices in V. We define an incidence relation by saying that two elements $x, y \in P_\nu$ are *incident* if there exist lattices $L \in x$ and $L' \in y$ such that

$$\pi L \subseteq L' \subseteq L$$

holds.

Remark 2.5.2. The incidence relation is reflexive and it is symmetric since for the two representatives L and L' we have

$$\pi L' \subseteq \pi L \subseteq L' \subseteq L.$$

Thus we can associate a flag complex $X_\nu = \Delta(P_\nu)$ to P_ν.

Definition 2.5.3. We define the *Bruhat-Tits building* $X = X_\nu$ associated to $\mathrm{PGL}_n(K_\nu)$ to be the flag complex X_ν in 2.5.2. It is the simplicial complex, where the vertices are the lattice classes of V corresponding to the valuation ν and a simplex of rank r is given by vertices $[L_0], ..., [L_{r-1}]$ such that

$$L_0 \supseteq L_1 \supseteq ... \supseteq L_{r-1} \supseteq \pi L_0$$

holds.

Proposition 2.5.4 ([AB08], Section 6.9.). *The Bruhat-Tits building X for $\mathrm{PGL}_n(K_\nu)$ is isomorphic to the Bruhat-Tits building $\Delta(B,N)$ for $SL_n(K_\nu)$, defined in 2.3.2. The fundamental chamber in X is the simplex with vertices $[\langle \pi e_1, ..., \pi e_{i-1}, e_i, ..., e_n\rangle]$, $i = 1, ..., n$.*

Remark 2.5.5. Let $[L]$ be a vertex of X. Then we can identify the neighbors of $[L]$ with subspaces in $L/\pi L \cong k_p^n$. Hence the link of $[L]$ is the spherical building associated to $\mathrm{SL}_n(k_p)$ as defined in 1.9.5.

2.5.1. The action of $\mathrm{GL}(V)$ and $\mathrm{PGL}(V)$

Remark 2.5.6. Since $\mathrm{GL}(V)$ acts on the set of bases of V, there is a natural action of $\mathrm{GL}(V)$ on the set of lattices of V and hence $\mathrm{GL}(V)$ acts on the vertices of X. This action preserves the incidence relation, so we get an action of $\mathrm{GL}(V)$ on X. Since the vertices are equivalence classes of lattices, where two lattices are equivalent if one is a scalar multiple of the other, the action of $\mathrm{GL}(V)$ induces an action of $\mathrm{PGL}(V)$ on X. Furthermore they have the same orbits.

Definition 2.5.7. Consider the Bruhat-Tits building X_ν corresponding to the valuation ν. Let L_s denote the standard lattice of V and define the *type* of the corresponding vertex $[L_s]$ in the Bruhat-Tits building to be 0. For each lattice L of V there exists an element $g \in \mathrm{GL}(V)$ such that $gL_s = L$. Define the *type* of $[L]$ as

$$\nu(det(g)) \pmod n.$$

Lemma 2.5.8. *The group $\mathrm{SL}(V)$ acts type preservingly and it acts transitive on the set of vertices of the same type.*

Proof. If $s \in \mathrm{SL}(V)$ and L is a lattice of V, then by definition the type of $s \cdot [L]$ is given by $\nu(det(s)) +$ type of $[L] \pmod n$. Thus $\mathrm{SL}(V)$ acts type preservingly, because $\nu(det(s)) = \nu(1) = 0$.
Let x and y be two vertices of the same type $i \in \{0, ..., n-1\}$. Then there exist two lattices L and L' in V such that $x = [L]$ and $y = [L']$. Furthermore there exist $g, g' \in \mathrm{GL}(V)$ with $gL_s = L$, $g'L_s = L'$. Therefore $g'g^{-1}L = L'$. Now

$$\nu(det(g'g^{-1})) = \nu(det(g')) + \nu(det(g^{-1})) = \nu(det(g')) - \nu(det(g)) \equiv i - i \equiv 0 \pmod n.$$

So there exist $m \in \mathbb{Z}$ with $n \cdot m = \nu(det(g'g^{-1}))$. Therefore $det(g'g^{-1}) = u \cdot \pi^{nm}$ for an uniformizer $\pi \in K$ and an unit element u in the valuation ring $\mathcal{O}_\nu$. Write $a = \pi^m$. Then it follows $det(g'g^{-1}) = ua^n$ and this yields, that $det(\frac{1}{a} \cdot g'g^{-1}) = u$. Now

$$s := \frac{1}{a} \begin{pmatrix} u^{-1} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \cdot g'g^{-1} \in \mathrm{SL}(V)$$

satisfies the condition $s \cdot L = \frac{1}{a} L'$. We conclude $s \cdot x = y$. $\square$

Remark 2.5.9. For a simplex in X given by the vertices $[L_0], ..., [L_r]$ we have that $L_i/\pi L_0$ is a subspace of the n-dimensional k_p-vector space $L_0/\pi L_0$ for all $0 \leq i \leq r$. Therefore the maximal simplices consist of n vertices. Moreover, we can find for a simplex of rank n a basis $(b_1, ..., b_n)$ of V such that

$$L_0 = \langle b_1, ..., b_n \rangle_{\mathcal{O}_\nu},\ L_1 = \langle \pi b_1, ..., b_n \rangle_{\mathcal{O}_\nu}, ...,\ L_{n-1} = \langle \pi b_1, ..., \pi b_{n-1}, b_n \rangle_{\mathcal{O}_\nu}.$$

Hence we find for all $0 \leq i \leq n-1$ a diagonal matrix $g_i \in \mathrm{GL}_n(K) \cong \mathrm{GL}(V)$ with $g_i L_0 = L_i$ such that $\nu(det(g_i)) = i$. This implies that all vertices of such a simplex are of different type.

2.5.2. Stabilizers

Proposition 2.5.10. *The stabilizer in* $\mathrm{GL}_n(K_\nu)$ *of the standard lattice* L_s *is* $\mathrm{GL}_n(\mathcal{O}_\nu)$ *and the stabilizer in* $\mathrm{GL}_n(K_\nu)$ *of the vertex* $[L_s]$ *corresponding to the standard lattice in* X *is* $Z \cdot \mathrm{GL}_n(\mathcal{O}_\nu)$*, where* $Z = \{z \cdot id \mid z \in K^\times\}$.

Proof. Let $g = (g_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} \in \mathrm{GL}_n(K)$ with $gL_s = L_s$. So we have

$$g\langle e_1, ..., e_n \rangle_{\mathcal{O}_\nu} = \langle e_1, ..., e_n \rangle_{\mathcal{O}_\nu}.$$

Therefore we have for all $j \in \{1, ..., n\}$:

$$ge_j = \sum_{i=1}^{n} g_{ij} e_i \in \langle e_1, ..., e_n \rangle_{\mathcal{O}_\nu}.$$

This implies $g_{ij} \in \mathcal{O}_\nu$ and since the determinant of g has to be an element in $K^\times \cap \mathcal{O}_\nu = \mathcal{O}_\nu^\times$ we derive $g \in \mathrm{GL}_n(\mathcal{O}_\nu)$. Since $\mathrm{GL}_n(\mathcal{O}_\nu)$ stabilizes the standard lattice, we conclude that $\mathrm{GL}_n(\mathcal{O}_\nu)$ is equal to the stabilizer of L_s.

Now suppose $g \in \mathrm{GL}_n(K)$ satisfies $g[L_s] = [L_s]$. Then there exists an element $a \in K^\times$ with $gL_s = aL_s$. Thus $a^{-1}gL_s = L_s$ which implies $a^{-1}g \in \mathrm{GL}_n(K)_{L_s} = \mathrm{GL}_n(\mathcal{O}_\nu)$. Hence we have $g \in a\,\mathrm{GL}_n(\mathcal{O}_\nu) \subseteq Z \cdot \mathrm{GL}_n(\mathcal{O}_\nu)$. With the observation that $Z \cdot \mathrm{GL}_n(\mathcal{O}_\nu)$ stabilizes $[L_s]$ we find that the stabilizer of the vertex corresponding to the standard lattice is equal to $Z \cdot \mathrm{GL}_n(\mathcal{O}_\nu)$. $\square$

Corollary 2.5.11. *The stabilizer in* $\mathrm{PGL}_n(K)$ *of the vertex corresponding to the standard lattice is* $\mathrm{PGL}_n(\mathcal{O}_\nu)$.

Proof. Follows from 2.5.10 $\square$

Remark 2.5.12. The fundamental apartment of the Bruhat-Tits building for $\mathrm{PGL}_n(K_\nu)$ has the vertices

$$[\langle \pi^{a_1} e_1, ..., \pi^{a_n} e_n \rangle],$$

where $(e_1, ..., e_n)$ denotes the standard basis of K^n and $a_1, ..., a_n \in \mathbb{Z}$ (cf. [AB08], Example 10.1.7). The stabilizer of the fundamental apartment is the monomial group. An apartment corresponds to a basis of K^n, in particular, the vertices of an apartment are of the form

$$[\langle \pi^{a_1} b_1, ..., \pi^{a_n} b_n \rangle]$$

for a basis $(b_1, ..., b_n)$ of K^n and $a_1, ..., a_n \in \mathbb{Z}$.

Definition 2.5.13. Let

$$\mathrm{GL}(V)^o = \{g \in \mathrm{GL}(V) \mid \nu(det(g)) = 0\}.$$

For a basis $(b_1, ..., b_n)$ of V, integers $a_1, ..., a_n \in \mathbb{Z}$ and two lattices $L_1 = \langle b_1, ..., b_n \rangle_{\mathcal{O}_\nu}$ and $L_2 = \langle \pi^{a_1} b_1, ..., \pi^{a_n} b_n \rangle_{\mathcal{O}_\nu}$ of V we define

$$\chi(L_1, L_2) = \sum_{i=1}^{n} a_i.$$

Remark 2.5.14. If we have two lattices L_1 and L_2 of V, then there exists an integer $m \in \mathbb{Z}$ such that $\pi^m L_2 \subseteq L_1$. By the invariant factor theorem there exists a basis $(b_1, ..., b_n)$ of V such that $L_1 = \langle b_1, ..., b_n \rangle_{\mathcal{O}_\nu}$ and $\pi^m L_2 = \langle \pi^{a'_1} b_1, ..., \pi^{a'_n} b_n \rangle_{\mathcal{O}_\nu}$ for uniquely determined integers $a'_1, ..., a'_n \in \mathbb{Z}$. This implies $L_2 = \langle \pi^{a'_1 - m} b_1, ..., \pi^{a'_n - m} b_n \rangle_{\mathcal{O}_\nu} = \langle \pi^{a_1} b_1, ..., \pi^{a_n} b_n \rangle_{\mathcal{O}_\nu}$ with $a_i = a'_i - m$ for $1 \leq i \leq n$. Moreover, the integers $\{a_1, ..., a_n\}$ does not depend on the choice of basis for the lattices L_1 and L_2.

Lemma 2.5.15 (see [Ser80], Chapter 2, Section 1.2, Proposition 1)**.** *Let L be a lattice and $s \in \mathrm{GL}(V)$. Then $\chi(L, s \cdot L) = \nu(det(s))$.*

Proof. Due to 2.5.14 we can find a basis $(b_1, ..., b_n)$ of the lattice L and integers $a_1, ..., a_n$ such that $(b_1 \pi^{a_1}, ..., b_n \pi^{a_n})$ is a basis of the lattice $s \cdot L$. Thus there exist for $1 \leq i, j \leq n$ elements $m_{ij} \in \mathcal{O}_\nu$ with $s \cdot b_j = \sum_{i=1}^n m_{ij} \pi^{a_i} b_i$ for $1 \leq j \leq n$. Hence

$$s = \begin{pmatrix} \pi^{a_1} & & \\ & \ddots & \\ & & \pi^{a_n} \end{pmatrix} \cdot \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{pmatrix}, \text{ where } \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{pmatrix} \in \mathrm{GL}_n(\mathcal{O}_\nu).$$

With $\nu\left(det\left(\begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{pmatrix}\right)\right) = 0$ it follows

$$\nu(det(s)) = \nu\left(det\left(\begin{pmatrix} \pi^{a_1} & & \\ & \ddots & \\ & & \pi^{a_n} \end{pmatrix}\right)\right) + \nu\left(det\left(\begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{pmatrix}\right)\right) =$$

$\sum_{i=1}^n a_i$. With $\chi(L, s \cdot L) = \sum_{i=1}^n a_i$ we proved the Lemma. $\square$

Lemma 2.5.16 (see [Ser80], Chapter 2, Section 1.3, Lemma 1). *For all subgroups $G \leq \mathrm{GL}(V)^o$ and lattices L of V we have $G_{[L]} = G_L$.*

Proof. Let g be an element in G with $g \cdot [L] = [L]$. Then there exist $c \in K^\times$ such that $gL = cL$. With $\chi(L, c \cdot L) = n \cdot \nu(c)$ and the previous Lemma 2.5.15 it follows: $\nu(det(g)) = \chi(L, cL) = n \cdot \nu(c)$. Because of $\nu(det(g)) = 0$ we deduce $\nu(c) = 0$, which means $c \in \mathcal{O}_\nu^\times$. Thus $L = c \cdot L$. □

Proposition 2.5.17 ([AB08], Section 6.9.). *The stabilizer in $SL_n(K_\nu)$ of the fundamental chamber C in $\Delta(B, N)$ is the intersection of the stabilizers of all of its vertices, in particular, it equals B.*

2.6. The two dimensional Bruhat-Tits building for PGL_3

From now on we focus on the case PGL_3. So we assume that V is a 3-dimensional vector space over a field K with discrete valuation ν. We first define a distance in the underlying graph of the Bruhat-Tits building (see the first subsection 2.6.1 below). In the second subsection we consider the case where $K = k(t)$ is the rational function field over a finite field $k = \mathbb{F}_q$ with q elements. There we compute a fundamental domain for the action of $\mathrm{GL}_n(k[t])$ on the underlying graph of the Bruhat-Tits building corresponding to the place ∞ of $k(t)$.

2.6.1. Distance in the underlying graph

We generalize the definition of distance in the Bruhat-Tits tree given in Chapter 2, Section 1.1 in [Ser80].

Let L and L' be two lattices of V. According to 2.5.14 we can find an $\mathcal{O}_\nu$-basis (b_1, b_2, b_3) of L and integers a, b, c such that $(\pi^a b_1, \pi^b b_2, \pi^c b_3)$ is an $\mathcal{O}_\nu$-basis for L'. Moreover, the integers $\{a, b, c\}$ does not depend on the choice of basis for L and L'. The lattice L' is a subset of L if and only if the three integers a, b and c are all not negative. For $L' \subset L$ we obtain

$$L/L' \cong \mathcal{O}_\nu/\pi^a\mathcal{O}_\nu \oplus \mathcal{O}_\nu/\pi^b\mathcal{O}_\nu \oplus \mathcal{O}_\nu/\pi^c\mathcal{O}_\nu.$$

Definition 2.6.1. Let x and y be two vertices of the underlying graph X of the Bruhat-Tits building and take lattices $L = \mathcal{O}_\nu b_1 \oplus \mathcal{O}_\nu b_2 \oplus \mathcal{O}_\nu b_3$ in x and $L' = \pi^a\mathcal{O}_\nu b_1 \oplus \pi^b\mathcal{O}_\nu b_2 \oplus \pi^c\mathcal{O}_\nu b_3$ in y. We define the *distance* in X of x and y to be the integer

$$d(x, y) = |max\{a, b, c\} - min\{a, b, c\}|.$$

Remark 2.6.2. a) The distance is independent of the choice of representatives for x and y, because if we replace L by rL and L' by sL' for some $r, s \in K^\times$, then we have to replace the integers $\{a, b, c\}$ by $\{a+\nu(\frac{s}{r}), b+\nu(\frac{s}{r}), c+\nu(\frac{s}{r})\}$ and hence the distance does not change.

b) If a lattice L is given, then each class $y \in X$ has exactly one representative L' satisfying the equivalent conditions:

- $L' \subseteq L$ and L' is maximal (in y) with this property
- $L' \subseteq L$ and $L' \nsubseteq L\pi$

For such a lattice L' we have

$$L/L' \cong \mathcal{O}_\nu/\pi^a\mathcal{O}_\nu \oplus \mathcal{O}_\nu/\pi^b\mathcal{O}_\nu$$

for some suitable non-negative integers a and b. In this case the distance between the lattice classes of L and L' is given by the maximum of a and b.

c) It is $d(x, y) = 0$ if and only if $x = y$. The vertices x and y are adjacent, i.e. $d(x, y) = 1$, if and only if there exist representatives $L \in x$ and $L' \in y$ with $L' \subseteq L$ and $L/L' \cong \mathcal{O}_\nu/\pi\mathcal{O}_\nu = k_p$.

Remark 2.6.3. Take a vertex x and a lattice $L \in x$, then $L/\pi L$ is a free k_p-module of rank 3. If we have another vertex y with $d(x, y) = 1$, then we can find a representative $L' \in y$ with $L' \subseteq L$ and $L'/\pi L$ is a k_p-submodule of $L/\pi L$ of rank 1 or rank 2. Therefore the edges with origin x correspond bijectively to the points and lines in the projective plane $\mathbb{P}_2(\mathcal{O}_\nu/\pi\mathcal{O}_\nu) = \mathbb{P}_2(k_p)$. For a finite field $k_p = \mathbb{F}_q$ we have $2|\mathbb{P}_2(k_p)| = 2(q^2 + q + 1)$ such edges.

Remark 2.6.4. There is another way to compute the distance between two vertices x and y in X with representatives $L \in x$ and $L' \in y$: Let $r \in \mathbb{Z}$ be minimal such that $\pi^r L \subseteq L'$ and $s \in \mathbb{Z}$ maximal according to the property $L' \subseteq \pi^s L$. Then the distance is given by

$$d(x, y) = |r - s|.$$

When we choose an $\mathcal{O}_\nu$-basis (b_1, b_2, b_3) of L such that $(\pi^a b_1, \pi^b b_2, \pi^c b_3)$ is an $\mathcal{O}_\nu$-basis for L' for suitable $a, b, c \in \mathbb{Z}$, then $r = max\{a, b, c\}$ and $s = min\{a, b, c\}$.

2.6.2. The fundamental domain

In this section we generalize the arguments given in [Ser80], Chapter 2. The definition of a fundamental domain can be found in Chapter 1, Section 4.1, Definition 7 in [Ser80].

Let $k = \mathbb{F}_q$ be the finite field with q elements and ∞ the place corresponding to the valuation $\nu_\infty : k(t) \to \mathbb{Z} \cup \{\infty\}$ defined by $\nu_\infty\left(\frac{a}{b}\right) = deg(b) - deg(a)$ for $a, b \in k[t]$ and

$a \neq 0 \neq b$ and $\nu_\infty(0) = \infty$. Then t^{-1} is a uniformizer. Let X be the underlying graph of the Bruhat-Tits building corresponding to the place ∞. Then the vertices of X are given by lattice classes in $k(t)^3$. Two vertices are adjacent if and only if there exist representatives L and L' for the lattice classes such that $t^{-1}L \subset L' \subset L$.

Consider now the action of $\Gamma = \mathrm{GL}_3(k[t]) \cong \mathrm{GL}_3(\mathcal{O}_{\{\infty\}})$ on X.

Definition 2.6.5. For all natural numbers $n \geq m \geq 0$ let $L_{n,m}$ denote the lattice $t^n\mathcal{O}_\infty e_1 \oplus t^m\mathcal{O}_\infty e_2 \oplus \mathcal{O}_\infty e_3$, where $\{e_1, e_2, e_3\}$ is the standard basis of the vector space $k(t)^3$.

The next Propositions generalize Proposition 3 in Chapter 1, Section 1.6 in the book [Ser80].

Proposition 2.6.6. *The vertices $[L_{n,m}]$ for $n \geq m \geq 0$ are pairwise inequivalent* mod Γ.

Proof. Let $s = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in \Gamma$ such that $s[L_{n,m}] = [L_{n+\alpha,m+\beta}]$. Hence there exist a $z \in \mathbb{Z}$ such that $s \cdot L_{n,m} = L_{n+\alpha,m+\beta} \cdot t^{-z}$, i.e. we have

$$\begin{gathered} at^n\mathcal{O}_\nu e_1 \oplus dt^n\mathcal{O}_\nu e_2 \oplus gt^n\mathcal{O}_\nu e_3 \in \langle t^{n+\alpha-z}e_1, t^{m+\beta-z}e_2, t^{-z}e_3\rangle_{\mathcal{O}_\nu}, \\ bt^m\mathcal{O}_\nu e_1 \oplus et^m\mathcal{O}_\nu e_2 \oplus ht^m\mathcal{O}_\nu e_3 \in \langle t^{n+\alpha-z}e_1, t^{m+\beta-z}e_2, t^{-z}e_3\rangle_{\mathcal{O}_\nu}, \\ c\mathcal{O}_\nu e_1 \oplus f\mathcal{O}_\nu e_2 \oplus i\mathcal{O}_\nu e_3 \in \langle t^{n+\alpha-z}e_1, t^{m+\beta-z}e_2, t^{-z}e_3\rangle_{\mathcal{O}_\nu}. \end{gathered}$$

We deduce: $deg(a) \leq \alpha - z$, $deg(b) \leq n + \alpha - z - m$, $deg(c) \leq n + \alpha - z$, $deg(d) \leq m + \beta - z - n$, $deg(e) \leq \beta - z$, $deg(f) \leq m + \beta - z$, $deg(g) \leq -n - z$, $deg(h) \leq -m - z$, $deg(i) \leq -z$. Because s is in $\mathrm{GL}_3(k[t])$ the determinant of s is a unit in the ring $k[t]$. It is $k[t]^\times = k^\times$ and this yields $\nu_\infty(det(s)) = 0$. Lemma 2.5.15 implies: $0 = \nu_\infty(det(s)) = \chi(L_{n,m}, s \cdot L_{n,m}) = \alpha + \beta - 3z$, i.e.

$$3z = \alpha + \beta.$$

Now we want to show, that z equals to 0:

Assume $z > 0$. Then $deg(g) < 0$, $deg(h) < 0$, $deg(i) < 0$, i.e. $g = h = i = 0$, a contradiction to $s \in \Gamma$.

Assume $z < 0$. Consider the matrix $s^{-1} = \begin{pmatrix} \tilde{a} & \tilde{b} & \tilde{c} \\ \tilde{d} & \tilde{e} & \tilde{f} \\ \tilde{g} & \tilde{h} & \tilde{i} \end{pmatrix} \in \Gamma$.

From $s \cdot L_{n,m} = L_{n+\alpha,m+\beta} \cdot t^{-z}$ it follows $L_{n,m} = s^{-1} \cdot L_{n+\alpha,m+\beta} \cdot t^{-z}$. Thus

$$\begin{gathered} \tilde{a}t^{n+\alpha-z}\mathcal{O}_\nu e_1 \oplus \tilde{d}t^{n+\alpha-z}\mathcal{O}_\nu e_2 \oplus \tilde{g}t^{n+\alpha-z}\mathcal{O}_\nu e_3 \in \langle t^n e_1, t^m e_2, e_3\rangle_{\mathcal{O}_\nu}, \\ \tilde{b}t^{m+\beta-z}\mathcal{O}_\nu e_1 \oplus \tilde{e}t^{m+\beta-z}\mathcal{O}_\nu e_2 \oplus \tilde{h}t^{m+\beta-z}\mathcal{O}_\nu e_3 \in \langle t^n e_1, t^m e_2, e_3\rangle_{\mathcal{O}_\nu}, \\ \tilde{c}t^{-z}\mathcal{O}_\nu e_1 \oplus \tilde{f}t^{-z}\mathcal{O}_\nu e_2 \oplus \tilde{i}t^{-z}\mathcal{O}_\nu e_3 \in \langle t^n e_1, t^m e_2, e_3\rangle_{\mathcal{O}_\nu}. \end{gathered}$$

Therefore $deg(\tilde{a}) \leq z-\alpha$, $deg(\tilde{b}) \leq n-m-\beta+z$, $deg(\tilde{c}) \leq n+z$, $deg(\tilde{d}) \leq m-n-\alpha+z$, $deg(\tilde{e}) \leq z-\beta$, $deg(\tilde{f}) \leq m+z$, $deg(\tilde{g}) \leq z-n-\alpha$, $deg(\tilde{h}) \leq z-m-\beta$, $deg(\tilde{i}) \leq z$. The assumtion $z < 0$ yields with $deg(\tilde{g}) < 0$, $deg(\tilde{h}) < 0$, $deg(\tilde{i}) < 0$ a contradiction to $s^{-1} \in \Gamma$.

Hence $z = 0$ and $\alpha = -\beta$.

Next we want to show, that $\alpha = 0$ and hence also $\beta = 0$:

Assume $\alpha < 0$. If $n = 0$, then $deg(a) \leq \alpha < 0$, $deg(b) \leq \alpha < 0$, $deg(c) \leq \alpha < 0$, which contradicts $s \in \Gamma$. For $n > 0$ it follows with $m + \beta \leq n + \alpha$ that $deg(g) \leq -n < 0$, $deg(d) \leq m + \beta - n \leq n + \alpha - n = \alpha < 0$ and $deg(a) \leq \alpha < 0$. This is again a contradiction to $s \in \Gamma$.

Now assume that $\alpha > 0$. Then, by similar arguments for the inverse of s, we have $deg(\tilde{a}) \leq -\alpha < 0$, $deg(\tilde{d}) \leq m - n - \alpha < 0$ and $deg(\tilde{g}) \leq -n - \alpha < 0$, which contradicts $s^{-1} \in \Gamma$.

It follows $\alpha = 0$ and $\beta = -\alpha = 0$. □

Next we define certain subgroups of $\mathrm{GL}_3(\mathbb{F}_q[t])$.

Definition 2.6.7. First let $H_{0,0} := \mathrm{GL}_3(\mathbb{F}_q)$.

For $n > m > 0$ define

$$H_{n,m} := \left\{ \begin{pmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{pmatrix} \mid a, e, i \in \mathbb{F}_q^\times, deg(b) \leq n-m, deg(c) \leq n, deg(f) \leq m \right\}.$$

Furthermore, define for $n > 0$ the groups

$$H_{n,n} := \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & i \end{pmatrix} \mid \begin{pmatrix} a & b \\ d & e \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_q), i \in \mathbb{F}_q^\times, deg(c) \leq n, deg(f) \leq n \right\}$$

and

$$H_{n,0} := \left\{ \begin{pmatrix} a & b & c \\ 0 & e & f \\ 0 & h & i \end{pmatrix} \mid \begin{pmatrix} e & f \\ h & i \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_q), a \in \mathbb{F}_q^\times, deg(b) \leq n, deg(c) \leq n \right\}.$$

Remark 2.6.8. For the groups $H_{n,m}$ (now considered as subgroups of $\mathrm{PGL}_3(k(t))$, because in Chapter 3 where we need the facts listed here we work in the projective general linear group) we have the following cardinalities:

If $n = m = 0$ it is

$$|H_{0,0}| = |\,\mathrm{PGL}_3(\mathbb{F}_q)| = (q^3-1)(q^3-q)q^2,$$

if $n > m > 0$ we get

$$|H_{n,m}| = (q-1)^2 q^{2n+3},$$

if $n = m > 0$ then

$$|H_{n,n}| = (q^2-1)(q^2-q)q^{2n+2}$$

and if $n > m = 0$ we compute

$$|H_{n,0}| = (q^2 - 1)(q^2 - q)q^{2n+2}.$$

Note that we divided by $q - 1$ since we are working in the projective group.

Proposition 2.6.9. *The group $H_{n,m}$ is the stabilizer of the vertex $[L_{n,m}]$ in Γ for all $n \geq m \geq 0$.*

Proof. Consider again the matrix s and the degree restraints of the entries of s like in the proof of Proposition 2.6.6. We know that $\alpha = \beta = z = 0$.

If $n = 0$, it follows $m = 0$ and hence the inequalities $deg(a) \leq 0,\ deg(b) \leq 0,\ deg(c) \leq 0,\ deg(d) \leq 0,\ deg(e) \leq 0,\ deg(f) \leq 0,\ deg(g) \leq 0,\ deg(h) \leq 0,\ deg(i) \leq 0$, which means $s \in \mathrm{GL}_3(\mathbb{F}_q)$.

Now let $n > 0$.
For $n > m > 0$ we get $deg(a) \leq 0,\ deg(b) \leq n - m,\ deg(c) \leq n,\ deg(d) \leq m - n < 0,\ deg(e) \leq 0,\ deg(f) \leq m,\ deg(g) \leq -n < 0,\ deg(h) \leq -m < 0,\ deg(i) \leq 0$, which yields $s \in H_{n,m}$.
For the cases $n = m$, respective $m = 0$, we get similarly that s is in $H_{n,n}$, respective $H_{n,0}$. ☐

Proposition 2.6.10. *a) $H_{0,0}$ acts with two orbits on the set of edges with origin $[L_{0,0}]$.*

b) For $n > 0$, $H_{n,n}$ fixes the edge $([L_{n,n}], [L_{n+1,n+1}])$ and acts with three orbits on the set of edges with origin $[L_{n,n}]$ distinct from $([L_{n,n}], [L_{n+1,n+1}])$.

c) For $n > 0$, $H_{n,0}$ fixes the edge $([L_{n,0}], [L_{n+1,0}])$ and acts with three orbits on the set of edges with origin $[L_{n,0}]$ distinct from $([L_{n,0}], [L_{n+1,0}])$.

d) For $n > m > 0$, $H_{n,m}$ fixes the edges $([L_{n,m}], [L_{n+1,m}])$ and $([L_{n,m}], [L_{n+1,m+1}])$ and acts with four orbits on the set of edges with origin $[L_{n,m}]$ distinct from $([L_{n,m}], [L_{n+1,m}])$ and $([L_{n,m}], [L_{n+1,m+1}])$.

Proof. a) The set of edges with origin $[L_{0,0}]$ can be identified with the set of one-dimensional and two-dimensional subspaces of k^3 (cf. Remark 2.6.3). Now the group $H_{0,0} = \mathrm{GL}_3(k)$ acts transitive on both, the set of one-dimensional subspaces and the set of two-dimensional subspaces. Thus $H_{0,0}$ acts with two orbits on the set of edges with origin $[L_{0,0}]$.

b) First note that $H_{n,n} \subset H_{n+1,n+1}$, which shows that $H_{n,n}$ fixes the edge $([L_{n,n}], [L_{n+1,n+1}])$. The action of $H_{n,n}$ on the k vector space $L_{n,n}/t^{-1}L_{n,n}$ is given by the homomorphism

$$H_{n,n} \to \mathrm{GL}_3(k), \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & i \end{pmatrix} \mapsto \begin{pmatrix} a & b & c_n \\ d & e & f_n \\ 0 & 0 & i \end{pmatrix},$$

where $c = c_0 + c_1 t + ... + c_n t^n$ and $f = f_0 + f_1 t + ... + f_n t^n$ with $c_i, f_i \in k$ for all $0 \leq i \leq n$. Therefore the action of $H_{n,n}$ on $L_{n,n}/t^{-1}L_{n,n} \cong k^3$ may be described by the action of the image of this homomorphism on the vector space k^3. Now the image is given by the subgroup of $\mathrm{GL}_3(k)$ consisting of matrices of the form

$$\begin{pmatrix} \star & \star & \star \\ \star & \star & \star \\ 0 & 0 & \star \end{pmatrix}.$$

This subgroup acts with two orbits on the set of one-dimensional subspaces of k^3, it fixes the subspace $\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$ and acts transitively on the set of two-dimensional subspaces of k^3 distinct from $\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$.

c) It is $H_{n,0}$ contained in $H_{n+1,0}$, hence $H_{n,0}$ fixes the edge $([L_{n,0}], [L_{n+1,0}])$. Similarly to the proof above the action of $H_{n,0}$ on $L_{n,0}/t^{-1}L_{n,0} \cong k^3$ is given by the homomorphism

$$H_{n,0} \to \mathrm{GL}_3(k), \begin{pmatrix} a & b & c \\ 0 & e & f \\ 0 & h & i \end{pmatrix} \mapsto \begin{pmatrix} a & b_n & c_n \\ 0 & e & f \\ 0 & h & i \end{pmatrix},$$

where $b = b_0 + b_1 t + ... + b_n t^n$ and $c = c_0 + c_1 t + ... + c_n t^n$ with $b_i, c_i \in k$ for all $0 \leq i \leq n$. The image of this homomorphism is the subgroup of matrices of the form

$$\begin{pmatrix} \star & \star & \star \\ 0 & \star & \star \\ 0 & \star & \star \end{pmatrix}.$$

This subgroup stabilizes the subspace $\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$ and acts transitively on the set of one-dimensional subspaces in k^3 distinct from $\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$. On the set of two-dimensional subspaces of k^3 this group acts with two orbits.

d) It is $H_{n,m}$ contained in $H_{n+1,m}$ and in $H_{n+1,m+1}$, hence $H_{n,m}$ fixes the edges $([L_{n,m}], [L_{n+1,m}])$ and $([L_{n,m}], [L_{n+1,m+1}])$. Consider the action of $H_{n,m}$ on

$L_{n,m}/t^{-1}L_{n,m} \cong k^3$ given by the homomorphism

$$H_{n,m} \to \mathrm{GL}_3(k), \begin{pmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{pmatrix} \mapsto \begin{pmatrix} a & b_{n-m} & c_n \\ 0 & e & f_m \\ 0 & 0 & i \end{pmatrix},$$

where $b = b_0 + b_1 t + ... + b_{n-m}t^{n-m}$, $c = c_0 + c_1 t + ... + c_n t^n$ and $f = f_0 + f_1 t + ... + f_m t^m$ with $b_i, c_j, f_r \in k$ for all $0 \leq i \leq n-m$, $0 \leq j \leq n$, $0 \leq r \leq m$. The image of this homomorphism is the subgroup of triangular matrices

$$\begin{pmatrix} \star & \star & \star \\ 0 & \star & \star \\ 0 & 0 & \star \end{pmatrix}.$$

This subgroup stabilizes the subspace $\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$ and acts with two orbits on the set of one-dimensional subspaces in k^3 distinct from $\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$. Furthermore it fixes $\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$ and acts with two orbits on the set of two-dimensional subspaces of k^3 distinct from $\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$.

□

Definition 2.6.11. Let Y be a graph, G be a group acting without inversion on Y (this means there does not exist an element $g \in G$ and an edge (x, y) of Y with $g(x, y) = (y, x)$) and $\pi : Y \to G\backslash Y$ be the canonical projection. A subgraph F of Y is called *fundamental domain* for the action of G on Y if the restriction $\pi|_F : F \to G\backslash Y$ is an isomorphism.

Theorem 2.6.12. *The set $F := \{[L_{n,m}] \mid n \geq m \geq 0\}$ of vertices of X is a fundamental domain for the action of Γ on X.*

Proof. We generalize the proof of the Corollary of Proposition 3 in Chapter 1, Section 1.6 in [Ser80]: First we have $\Gamma \subset \mathrm{GL}(k(t)^3)^o$, because for every matrix $M \in \Gamma$ the determinant of M is an element in $k[t]^\times = k^\times$, i.e. $\nu(det(M)) = 0$. Thus Γ acts without inversions on X and the quotient graph $X' = \Gamma\backslash X$ is defined. Because of Proposition 2.6.6 we know that the vertices in F are pairwiese inequivalent $\mod \Gamma$. Therefore the restriction of the projection $X \to X'$ to F yields an isomorphism from F to a subgraph F' in X'.

Now we want to show that F' equals X': X' is a quotient graph of the underlying graph X of a building, hence connected. Therefore it suffices to show, that F' is open in X', i.e. every edge y' in X' with origin in F' is contained in F'. Suppose y' is an edge in X', such that its origin p is a vertex in F'. Then there exists a vertex $[L_{n,m}]$ in F such that p is the image of the vertex $[L_{n,m}]$ under the projection $X \to X'$. Furthermore there exists an edge y in X with origin $[L_{n,m}]$, such that y' is the image of y under the projection $X \to X'$.

For $n = 0 = m$ it follows with Proposition 2.6.10 that y is congruent to

$$([L_{0,0}], [L_{1,0}]) \mod \Gamma \text{ or } ([L_{0,0}], [L_{1,1}]) \mod \Gamma$$

which yields y' is an edge in F'. Similarly we get from Proposition 2.6.10 that y is congruent to $z \mod \Gamma$, where z is as follows:
if $n > m = 0$, then

$$z \in \{([L_{n,0}], [L_{n+1,0}]), ([L_{n,0}], [L_{n-1,0}]), ([L_{n,0}], [L_{n,1}]), ([L_{n,0}], [L_{n+1,1}])\};$$

if $n = m > 0$, then

$$z \in \{([L_{n,n}], [L_{n+1,n}]), ([L_{n,n}], [L_{n+1,n+1}]), ([L_{n,n}], [L_{n,n-1}]), ([L_{n,n}], [L_{n-1,n-1}])\}$$

and if $n > m > 0$, then

$$z \in \{([L_{n,m}], [L_{n+1,m}]), ([L_{n,m}], [L_{n+1,m+1}]), ([L_{n,m}], [L_{n,m+1}]), ([L_{n,m}], [L_{n-1,m}]), ([L_{n,m}], [L_{n-1,m-1}]), ([L_{n,m}], [L_{n,m-1}])\}.$$

In all of these cases y' is an edge in F'. □

Remark 2.6.13. Due to [Sou79] we know even more. In particular, when we glue in a 2-simplex for every triangle in F, then the resulting sector forms a simplicial fundamental domain for the action of Γ on the Bruhat-Tits building X.

3. Quotient-graphs for certain subgroups of $\mathrm{PGL}_3(\mathbb{F}_q(t))$

In the following chapter we will state and prove our Main Theorem, which describes the quotient graph of the underlying graph of the Bruhat-Tits building by the action of an arithmetic subgroup of the projective group over a rational function field. In the first section we formulate the Theorem, what will take 9 pages. Then we will need the rest of the chapter to prove this Theorem. The strategy for this proof is the same as in [KMS15]: For a place p of the rational function field $\mathbb{F}_q(t)$ we consider the action of $\mathrm{PGL}_3(\mathcal{O}_{\{p,\infty\}})$ on the product of the Bruhat-Tits building associated to the place p and the Bruhat-Tits building associated to the place ∞. In order to describe the quotient graph of the underlying graph X of the Bruhat-Tits building corresponding to the place p by the action of $\Gamma = \mathrm{PGL}_3(\mathcal{O}_{\{p\}})$ we can use the fundamental domain for the action of $\mathrm{PGL}_3(\mathcal{O}_{\{\infty\}})$ computed in 2.6.12. The vertices $X_{n,m}$ in the orbit space $\Gamma\backslash X$ correspond to certain vertices in the fundamental domain. The number of edges between two given vertices $X_{n,m}$ and $X_{n',m'}$ in the orbit space $\Gamma\backslash X$ is equal to the number of double cosets $H_{n',m'} \setminus \Upsilon_{n',m',n,m}/H_{n,m}$, where $H_{n,m}$, respective $H_{n',m'}$, is the stabilizer of the vertex $y_{n,m}$, respective $y_{n',m'}$, in the Bruhat-Tits building associated to the place ∞. Moreover, $\Upsilon_{n',m',n,m}$ is the set of matrices in $\mathrm{PGL}_3(\mathcal{O}_{\{p,\infty\}})$ mapping $x_{0,0}$ to a neighbor and $y_{n,m}$ to $y_{n',m'}$. The size of a double coset $H_{n',m'}MH_{n,m}$ with representative $M \in \Upsilon_{n',m',n,m}$ is given by $|H_{n',m'}MH_{n,m}| = \frac{|H_{n',m'}||H_{n,m}|}{|M^{-1}H_{n',m'}M \cap H_{n,m}|}$. Due to 2.6.8 we know the cardinalities of the two stabilizers. Hence we have to compute the size $|M^{-1}H_{n',m'}M \cap H_{n,m}|$ to find the length of a double coset. Furthermore we have to count the elements in the set $\Upsilon_{n',m',n,m}$ to obtain the number of double cosets in a given case. We first compute the cardinality of $\Upsilon_{n',m',n,m}$ for the needed cases. Then we distinguish three cases ($\kappa := \frac{d-2n-n'+m-m'}{3} < 0, \kappa = 0$ and $\kappa > 0$) to calculate the number of double cosets. The cases follow from the degree restraint for the lower left entry of the representative M, in particular for $\kappa < 0$ we know that this entry has to be zero, in case $\kappa = 0$ it is an element in the field $\mathbb{F}_q$ and for $\kappa > 0$ we only know that it is a polynomial with degree less or equal to κ. Since we have four different types of stabilizers $H_{n,m}$ we have to consider four subcases for n, m, respective n', m', which yields in total 16 subcases in all three cases.

3.1. Main Theorem

Theorem 3.1.1. *Let ν_p be a valuation of degree d of the rational function field $\mathbb{F}_q(t)$ and let X_{ν_p} be the Bruhat-Tits building of the group $\mathrm{PGL}_3(\mathbb{F}_q(t)_{\nu_p})$. Consider the underlying graph X of X_{ν_p}. Then the orbit space $\mathrm{PGL}_3(\mathcal{O}_{\{p\}})\backslash X$ can be described as follows.*

1. *If $3 \nmid d$, then its set of vertices is $\{X_{n,m} \mid n, m \in \mathbb{N}_0$ and $n \geq m \geq 0\}$ with the following edges, where $n, m \in \mathbb{N}_0$ and $i, j \in \mathbb{N}$:*

   a) 1 edge between the vertices

   i. $X_{\{n,m\}}$ *and* $X_{\{n+d,m\}}$; *here* $X_{\{n,m\}} := \begin{cases} X_{n,m} & \text{if } n \geq m \\ X_{m,n} & \text{if } m \geq n, \end{cases}$

   ii. $X_{n,m}$ *and* $X_{n+d,m+d}$ $\quad (n \geq m \geq 0)$,

   iii. $X_{n,m}$ *and* $X_{n-m+d,d-m}$ $\quad (n \geq m > 0, d > m)$ $\quad$ *if* $d \geq 2$,

   iv. $X_{n,m}$ *and* $X_{d-m,n-m}$ $\quad (d > n \geq m > 0)$ $\quad$ *if* $d \geq 2$,

   b) $q^{2d-2i-2}$ edges, if $d \geq 4$, between the vertices

   i. $X_{n,0}$ *and* $X_{n+i,2i-d}$ $\quad (2i > d > i > 1)$,

   ii. $X_{n+i,n+i}$ *and* $X_{2i+n-d,n}$ $\quad (2i > d > i > 1)$,

   c) $q^{2d-2n-2i-2}$ edges, if $d \geq 4$, between the vertices

   i. $X_{i,i}$ *and* $X_{2n+2i-d,n}$ $\quad (2i + n > d > i + n > 2)$,

   d) q^{2i-2} edges, if $d \geq 4$, between the vertices

   i. $X_{n,n}$ *and* $X_{d+n-i,n+i}$ $\quad (d > 2i)$,

   ii. $X_{n+i,0}$ *and* $X_{\{n,d-2i\}}$ $\quad (2i + n \neq d > 2i)$; *here* $X_{\{n,m\}} := \begin{cases} X_{n,m} & \text{if } n \geq m \\ X_{m,n} & \text{if } m \geq n, \end{cases}$

   e) $q^{2d-2n-2}$ edges, if $d \geq 4$, between the vertices

   i. $X_{n,0}$ *and* $X_{2n-d,2n-d}$ $\quad (2n > d > n > 2)$, *f)* q^{2n-2} *edges, if* $d \geq 4$*, between the vertices*

i. $X_{n,n}$ *and* $X_{d-2n,d-2n}$ $\quad (d > 2n > 0)$,

g) $q^{2d-2n-6}$ *edges between the vertices*

i. $X_{n,n}$ *and* $X_{0,0}$ $\quad (d > n > 0)$ $\quad$ *if* $d \equiv n \pmod 3$ *and* $d \geq 4$,
ii. $X_{n,0}$ *and* $X_{0,0}$ $\quad (d > n > 0)$ $\quad$ *if* $d \equiv n \pmod 3$ *and* $d \geq 4$,
iii. $X_{n,0}$ *and* $X_{0,0}$ $\quad (d > 2n > 0)$ $\quad$ *if* $d \equiv 2n \pmod 3$ *and* $d \geq 5$,
iv. $X_{n,n}$ *and* $X_{0,0}$ $\quad (d > 2n > 0)$ $\quad$ *if* $d \equiv 2n \pmod 3$ *and* $d \geq 5$,

h) $q^{d+\kappa-4}$ *edges, if* $d \geq 5$*, between the vertices*

i. $X_{n,0}$ *and* $X_{i,i}$ $\quad (n, -\kappa = -\frac{d-2i-2n}{3} \in \mathbb{N}, \quad d > 2n - i, d > 2i - n, 2n + 2i > d \geq 4 - \kappa)$,

i) $(q+1)q^{2d-2n-2}$ *edges, if* $d \geq 4$*, between the vertices*

i. $X_{n,2n-i-d}$ *and* $X_{i,i}$ $\quad (2n > d + i > n, d > n > 0)$,
ii. $X_{n,m}$ *and* $X_{n+m-d,0}$ $\quad (n + m > d > n > m > 0)$,

j) $(q+1)q^{2i-2j-2}$ *edges, if* $d \geq 4$*, between the vertices*

i. $X_{n+i,j}$ *and* $X_{\{n,d-2i+j\}}$ $\quad (i > j, d - 2i + j \neq n, d > 2i - j)$;

here $X_{\{n,m\}} := \begin{cases} X_{n,m} & \text{if } n \geq m \\ X_{m,n} & \text{if } m \geq n, \end{cases}$

k) $(q+1)q^{2n+2j-2i-2}$ *edges, if* $d \geq 5$*, between the vertices*

i. $X_{i,j}$ *and* $X_{i+d-2j-n,n}$ $\quad (i > j, n > 0, d > 2j + 2n - i \geq i + 2)$,

l) $(q+1)q^{2i-2}$ *edges, if* $d \geq 4$*, between the vertices*

i. $X_{n,m}$ *and* $X_{d+m-i,n+i}$ $\quad (n > m > 0, d > 2i + n - m)$, *m)* $(q+1)q^{2m-2}$ *edges, if* $d \geq 4$*, between the vertices*

i. $X_{n,m}$ *and* $X_{d-n-m,d-n-m}$ $(d > n+m > n > m > 0)$,

n) $(q+1)q^{2d-2n-6}$ *edges, if* $d \geq 7$*, between the vertices*

i. $X_{n,m}$ *and* $X_{0,0}$ $(n > m > 0, d > 2n - m)$ *if* $d \equiv 2n - m \pmod 3 \equiv 2(n+m) \pmod 3$,

ii. $X_{n,m}$ *and* $X_{0,0}$ $(n > m > 0, d > n + m)$ *if* $d \equiv n + m \pmod 3$,

o) $(q+1)q^{d+\kappa-m-4}$ *edges, if* $d \geq 7$*, between the vertices*

i. $X_{n,m}$ *and* $X_{i,i}$ $(-\kappa = -\frac{d-2n-2i+m}{3} \in \mathbb{N}, n > m > 0, d > 2n - m - i, d > 2i - n + 2m, d \geq m + 4 - \kappa)$,

p) $(q+1)q^{d+\kappa-n+m-4}$ *edges, if* $d \geq 7$*, between the vertices*

i. $X_{n,m}$ *and* $X_{i,0}$ $(-\kappa = -\frac{d-2i-n-m}{3} \in \mathbb{N}, n > m > 0, d > n + m - i, d > 2i + n - 2m, d \geq n - m + 4 - \kappa)$,

q) $(q+1)^2q^{d+\kappa-m+j-i-2}$ *edges, if* $d \geq 5$*, between the vertices*

i. $X_{n,m}$ *and* $X_{i,j}$ $(-\kappa = -\frac{d-2n-i+m-j}{3} \in \mathbb{N}, n > m > 0, i > j, d > i - n + 2m + j, d > 2n + i - m - 2j, d \geq m - \kappa + i - j + 2)$,

r) $(q+1)q^{2(d-n-i-2)} + q^{2(d-n-i-1)}$ *edges, if* $d \geq 5$*, between the vertices*

i. $X_{n,n}$ *and* $X_{i,2n-d+2i}$ $(2n + 2i > d > 2n + i > i, d \geq n + i + 2)$,

s) $(q+1)q^{2(n+i-2)} + q^{2(n+i-1)}$ *edges, if* $d \geq 5$*, between the vertices*

i. $X_{n,0}$ *and* $X_{d-2n-i,i}$ $(d > 2n + 2i > 2i)$,

t) $(q+1)^2q^{2(n-m+i-2)} + (q+1)q^{2(n-m+i-1)}$ *edges, if* $d \geq 7$*, between the vertices*

i. $X_{n,m}$ *and* $X_{d+m-2n-i,i}$ $(n > m > 0, d > 2n + 2i - m \geq m + 4)$, *u)* $(q+1)^2(q^2+q+1)q^{2(d-i-n-3)}$ *edges, if* $d \geq 10$*, between the vertices*

i. $X_{n,m}$ *and* $X_{i,j}$ $(n > m > 0, i > j, d > 2n+i+j-m)$ *if* $d \equiv 2n+i+j-m \pmod 3$,

v) $(q+1)(q^2+q+1)q^{2(d-i-n-3)}$ *edges between the vertices*

i. $X_{n,m}$ *and* $X_{i,i}$ $(n > m > 0, d > 2n+2i-m)$ *if* $d \equiv 2n+2i-m \pmod 3$ *and* $d \geq 8$,

ii. $X_{n,m}$ *and* $X_{i,0}$ $(n > m > 0, d > 2n+i-m)$ *if* $d \equiv 2n+i-m \pmod 3$ *and* $d \geq 7$,

iii. $X_{n,n}$ *and* $X_{i,j}$ $(i > j, d > n+i+j > i+j)$ *if* $d \equiv n+i+j \pmod 3$ *and* $d \geq 7$,

iv. $X_{n,0}$ *and* $X_{i,j}$ $(i > j, d > 2n+i+j > i+j)$ *if* $d \equiv 2n+i+j \pmod 3$ *and* $d \geq 8$,

w) $(q^2+q+1)q^{2(d-i-n-3)}$ *edges between the vertices*

i. $X_{n,n}$ *and* $X_{i,i}$ $(d > n+2i > 2i)$ *if* $d \equiv n+2i \pmod 3$ *and* $d \geq 7$,

ii. $X_{n,n}$ *and* $X_{i,0}$ $(d > n+i > i)$ *if* $d \equiv n+i \pmod 3$ *and* $d \geq 5$,

iii. $X_{n,0}$ *and* $X_{i,i}$ $(d > 2n+2i > 2i)$ *if* $d \equiv 2n+2i \pmod 3$ *and* $d \geq 7$,

iv. $X_{n,0}$ *and* $X_{i,0}$ $(d > 2n+i > i)$ *if* $d \equiv 2n+i \pmod 3$ *and* $d \geq 7$.

If d is additionally even:

x) 1 *edge, if* $d = 2$*, between the vertices*

i. $X_{n,n}$ *and* $X_{n+1,n+1}$,

ii. $X_{n,0}$ *and* $X_{n+1,0}$,

y) $\frac{q(q^{d-3}+1)}{q+1}$ *edges, if* $d \geq 4$*, between the vertices*

i. $X_{n,n}$ *and* $X_{n+\frac{d}{2},n+\frac{d}{2}}$,

ii. $X_{n,0}$ *and* $X_{n+\frac{d}{2},0}$,

z) $q^{d-4} + \frac{q(q^{d-3}+1)}{q+1}$ *edges, if* $d \geq 4$*, between the vertices*

i. $X_{n,n}$ *and* $X_{\frac{d-2n}{2},0}$ $(d > 2n > 0)$.

2. *If $3 \mid d$, then its set of vertices is* $\{X^r_{n,m} \mid n,m \in \mathbb{N}_0,\ n \geq m \geq 0,\ n+m \in 3\mathbb{N}_0$ *and* $r \in \mathbb{Z}_3\}$ *with the following edges, where* $n,m \in \mathbb{N}_0$ *and* $i,j \in \mathbb{N}$*:*

   a) *1 edge between the vertices*

      i. $X^r_{\{n,m\}}$ *and* $X^{r+1}_{\{n+d,m\}}$ $(n+m \in 3\mathbb{N}_0)$; *here* $X^r_{\{n,m\}} := \begin{cases} X^r_{n,m} & \text{if } n \geq m \\ X^r_{m,n} & \text{if } m \geq n, \end{cases}$

      ii. $X^{r+1}_{n,m}$ *and* $X^r_{n+d,m+d}$ $(n+m \in 3\mathbb{N}_0, n \geq m \geq 0)$,

      iii. $X^{r+1}_{n,m}$ *and* $X^r_{n-m+d,d-m}$ $(n+m \in 3\mathbb{N}, n \geq m > 0, d > m)$ *if* $d \geq 6$,

      iv. $X^r_{n,m}$ *and* $X^{r+1}_{d-m,n-m}$ $(n+m \in 3\mathbb{N}, d > n \geq m > 0)$ *if* $d \geq 6$,

   b) $q^{2d-2i-2}$ *edges between the vertices*

      i. $X^{r+1}_{n,0}$ *and* $X^r_{n+i,2i-d}$ $(n \in 3\mathbb{N}_0, 2i > d > i > 1)$,

      ii. $X^{r+1}_{n+i,n+i}$ *and* $X^r_{2i+n-d,n}$ $(n+i \in 3\mathbb{N}, 2i > d > i > 1)$,

   c) $q^{2d-2n-2i-2}$ *edges, if* $d \geq 6$*, between the vertices*

      i. $X^{r+1}_{i,i}$ *and* $X^r_{2n+2i-d,n}$ $(i \in 3\mathbb{N}, 2i+n > d > i+n > 2)$,

   d) q^{2i-2} *edges between the vertices*

      i. $X^r_{n,n}$ *and* $X^{r+1}_{d+n-i,n+i}$ $(n \in 3\mathbb{N}_0, d > 2i)$,

      ii. $X^r_{n+i,0}$ *and* $X^{r+1}_{\{n,d-2i\}}$ $(n+i \in 3\mathbb{N}, 2i+n \neq d > 2i)$;

      here $X^r_{\{n,m\}} := \begin{cases} X^r_{n,m} & \text{if } n \geq m \\ X^r_{m,n} & \text{if } m \geq n, \end{cases}$

   e) $q^{2d-2n-2}$ *edges, if* $d \geq 6$*, between the vertices*

      i. $X^r_{n,0}$ *and* $X^{r+1}_{2n-d,2n-d}$ $(n \in 3\mathbb{N}, 2n > d > n > 2)$, *f)* q^{2n-2} *edges, if* $d \geq 9$*, between the vertices*

i. $X^{r+1}_{n,n}$ *and* $X^{r}_{d-2n,d-2n}$ $\quad (n \in 3\mathbb{N}, d > 2n > 0)$,

g) $q^{2d-2n-6}$ *edges between the vertices*

i. $X^{r}_{n,n}$ *and* $X^{r+1}_{0,0}$ $\quad (n \in 3\mathbb{N}, d > n > 0)$ $\quad$ *if* $d \equiv n \pmod 3$ *and* $d \geq 6$,

ii. $X^{r+1}_{n,0}$ *and* $X^{r}_{0,0}$ $\quad (n \in 3\mathbb{N}, d > n > 0)$ $\quad$ *if* $d \equiv n \pmod 3$ *and* $d \geq 6$,

iii. $X^{r}_{n,0}$ *and* $X^{r+1}_{0,0}$ $\quad (n \in 3\mathbb{N}, d > 2n > 0)$ $\quad$ *if* $d \equiv 2n \pmod 3$ *and* $d \geq 9$,

iv. $X^{r+1}_{n,n}$ *and* $X^{r}_{0,0}$ $\quad (n \in 3\mathbb{N}, d > 2n > 0)$ $\quad$ *if* $d \equiv 2n \pmod 3$ *and* $d \geq 9$,

h) $q^{d+\kappa-4}$ *edges, if* $d \geq 6$*, between the vertices*

i. $X^{r}_{n,0}$ *and* $X^{r+1}_{i,i}$ $\quad (n, i \in 3\mathbb{N}, -\kappa = -\frac{d-2i-2n}{3} \in \mathbb{N}, \quad d > 2n - i, d > 2i - n, 2n + 2i > d \geq 4 - \kappa)$,

i) $(q+1)q^{2d-2n-2}$ *edges, if* $d \geq 6$*, between the vertices*

i. $X^{r}_{n,2n-i-d}$ *and* $X^{r+1}_{i,i}$ $\quad (i \in 3\mathbb{N}, 2n > d + i > n, d > n > 0)$,

ii. $X^{r+1}_{n,m}$ *and* $X^{r}_{n+m-d,0}$ $\quad (n + m \in 3\mathbb{N}, n + m > d > n > m > 0)$,

j) $(q+1)q^{2i-2j-2}$ *edges, if* $d \geq 6$*, between the vertices*

i. $X^{r}_{n+i,j}$ *and* $X^{r+1}_{\{n,d-2i+j\}}$ $\quad (n + i + j \in 3\mathbb{N}, i > j, d - 2i + j \neq n, d > 2i - j)$;

here $X^{r}_{\{n,m\}} := \begin{cases} X^{r}_{n,m} & \textit{if } n \geq m \\ X^{r}_{m,n} & \textit{if } m \geq n, \end{cases}$

k) $(q+1)q^{2n+2j-2i-2}$ *edges, if* $d \geq 6$*, between the vertices*

i. $X^{r}_{i,j}$ *and* $X^{r+1}_{i+d-2j-n,n}$ $\quad (i + j \in 3\mathbb{N}, i > j, n > 0, d > 2j + 2n - i \geq i + 2)$,

l) $(q+1)q^{2i-2}$ *edges, if* $d \geq 6$*, between the vertices*

i. $X^{r}_{n,m}$ *and* $X^{r+1}_{d+m-i,n+i}$ $\quad (n + m \in 3\mathbb{N}, n > m > 0, d > 2i + n - m)$, *m)* $(q+1)q^{2m-2}$ *edges, if* $d \geq 6$*, between the vertices*

i. $X^{r+1}_{n,m}$ *and* $X^{r}_{d-n-m,d-n-m}$ $\quad (n+m \in 3\mathbb{N}, d > n+m > n > m > 0)$,

n) $(q+1)q^{2d-2n-6}$ *edges, if* $d \geq 6$*, between the vertices*

i. $X^{r}_{n,m}$ *and* $X^{r+1}_{0,0}$ $\quad (n+m \in 3\mathbb{N}, n > m > 0, d > 2n-m)$ *if* $d \equiv 2(n+m) \pmod 3$,

ii. $X^{r+1}_{n,m}$ *and* $X^{r}_{0,0}$ $\quad (n+m \in 3\mathbb{N}, n > m > 0, d > n+m)$ $\quad$ *if* $d \equiv n+m \pmod 3$,

o) $(q+1)q^{d+\kappa-m-4}$ *edges, if* $d \geq 6$*, between the vertices*

i. $X^{r}_{n,m}$ *and* $X^{r+1}_{i,i}$ $\quad (n+m, i \in 3\mathbb{N}, -\kappa = -\frac{d-2n-2i+m}{3} \in \mathbb{N}, n > m > 0,$ $d > 2n-m-i, d > 2i-n+2m, d \geq m+4-\kappa)$,

p) $(q+1)q^{d+\kappa-n+m-4}$ *edges, if* $d \geq 6$*, between the vertices*

i. $X^{r+1}_{n,m}$ *and* $X^{r}_{i,0}$ $\quad (n+m, i \in 3\mathbb{N}, -\kappa = -\frac{d-2i-n-m}{3} \in \mathbb{N}, n > m > 0,$ $d > n+m-i, d > 2i+n-2m, d \geq n-m+4-\kappa)$,

q) $(q+1)^2 q^{d+\kappa-m+j-i-2}$ *edges, if* $d \geq 6$*, between the vertices*

i. $X^{r}_{n,m}$ *and* $X^{r+1}_{i,j}$ $\quad (n+m, i+j \in 3\mathbb{N}, -\kappa = -\frac{d-2n-i+m-j}{3} \in \mathbb{N}, n > m > 0, i > j,$ $d > i-n+2m+j, d > 2n+i-m-2j, d \geq m-\kappa+i-j+2)$,

r) $(q+1)q^{2(d-n-i-2)} + q^{2(d-n-i-1)}$ *edges, if* $d \geq 6$*, between the vertices*

i. $X^{r+1}_{n,n}$ *and* $X^{r}_{i,2n-d+2i}$ $\quad (n \in 3\mathbb{N}, 2n+2i > d > 2n+i > i, d \geq n+i+2)$,

s) $(q+1)q^{2(n+i-2)} + q^{2(n+i-1)}$ *edges, if* $d \geq 9$*, between the vertices*

i. $X^{r}_{n,0}$ *and* $X^{r+1}_{d-2n-i,i}$ $\quad (n \in 3\mathbb{N}, d > 2n+2i > 2i)$,

t) $(q+1)^2 q^{2(n-m+i-2)} + (q+1)q^{2(n-m+i-1)}$ *edges, if* $d \geq 6$*, between the vertices*

i. $X^{r}_{n,m}$ *and* $X^{r+1}_{d+m-2n-i,i}$ $\quad (n+m \in 3\mathbb{N}, n > m > 0, d > 2n+2i-m \geq m+4)$, *u)* $(q+1)^2(q^2+q+1)q^{2(d-i-n-3)}$ *edges, if* $d \geq 9$*, between the vertices*

i. $X^r_{n,m}$ *and* $X^{r+1}_{i,j}$ $(n+m, i+j \in 3\mathbb{N}, n > m > 0, i > j, d > 2n+i+j-m)$ *if* $d \equiv 2n+i+j-m \pmod 3$,

v) $(q+1)(q^2+q+1)q^{2(d-i-n-3)}$ *edges between the vertices*

i. $X^r_{n,m}$ *and* $X^{r+1}_{i,i}$ $(n+m, i \in 3\mathbb{N}, n > m > 0, d > 2n+2i-m)$ *if* $d \equiv 2n+2i-m \pmod 3$ *and* $d \geq 12$,

ii. $X^r_{n,m}$ *and* $X^{r+1}_{i,0}$ $(n+m, i \in 3\mathbb{N}, n > m > 0, d > 2n+i-m)$ *if* $d \equiv 2n+i-m \pmod 3$ *and* $d \geq 9$,

iii. $X^r_{n,n}$ *and* $X^{r+1}_{i,j}$ $(n, i+j \in 3\mathbb{N}, i > j, d > n+i+j > i+j)$ *if* $d \equiv n+i+j \pmod 3$ *and* $d \geq 9$,

iv. $X^r_{n,0}$ *and* $X^{r+1}_{i,j}$ $(n, i+j \in 3\mathbb{N}, i > j, d > 2n+i+j > i+j)$ *if* $d \equiv 2n+i+j \pmod 3$ *and* $d \geq 12$,

w) $(q^2+q+1)q^{2(d-i-n-3)}$ *edges between the vertices*

i. $X^r_{n,n}$ *and* $X^{r+1}_{i,i}$ $(n, i \in 3\mathbb{N}, d > n+2i)$ *if* $d \equiv n+2i \pmod 3$ *and* $d \geq 12$,

ii. $X^r_{n,n}$ *and* $X^{r+1}_{i,0}$ $(n, i \in 3\mathbb{N}, d > n+i)$ *if* $d \equiv n+i \pmod 3$ *and* $d \geq 9$,

iii. $X^r_{n,0}$ *and* $X^{r+1}_{i,i}$ $(n, i \in 3\mathbb{N}, d > 2n+2i)$ *if* $d \equiv 2n+2i \pmod 3$ *and* $d \geq 15$,

iv. $X^r_{n,0}$ *and* $X^{r+1}_{i,0}$ $(n, i \in 3\mathbb{N}, d > 2n+i)$ *if* $d \equiv 2n+i \pmod 3$ *and* $d \geq 12$.

If d *is additionally even:*

x) $\frac{q(q^{d-3}+1)}{q+1}$ *edges, if* $d \geq 6$*, between the vertices*

i. $X^r_{n,n}$ *and* $X^{r+1}_{n+\frac{d}{2},n+\frac{d}{2}}$ $(n \in 3\mathbb{N}_0)$,

ii. $X^{r+1}_{n,0}$ *and* $X^r_{n+\frac{d}{2},0}$ $(n \in 3\mathbb{N}_0)$,

y) $q^{d-4} + \frac{q(q^{d-3}+1)}{q+1}$ *edges, if* $d \geq 9$*, between the vertices*

i. $X^{r+1}_{n,n}$ *and* $X^r_{\frac{d-2n}{2},0}$ $(n \in 3\mathbb{N}, d > 2n)$.

The edges from $X_{0,0}$ to itself:

z) *Let* $l(s) := \begin{cases} \frac{d-2s-4}{3} & \text{if } s \equiv 1 \pmod 3 \\ \frac{d-2s-5}{3} & \text{if } s \equiv 2 \pmod 3 \\ \frac{d-2s-6}{3} & \text{if } s \equiv 0 \pmod 3 \end{cases}.$

For d is odd:

$$1+\frac{(q^{d-3}-1)(q^d+q^3-q^2-q)}{(q^3-1)(q^2-1)}-\frac{(q^d-q^3)(q^d+q^3)+q^d(q^d-q^3)}{(q^6-1)q^6}-(q+1)\left(\sum_{s=1}^{\frac{d-5}{2}} q^{2s}\frac{q^{6l(s)+6}-1}{q^6-1}\right)$$

edges, if $d \geq 9$, between the vertices

i. $X_{0,0}^r$ *and* $X_{0,0}^{r+1}$,

and 1 *edge, if $d = 3$, between the vertices*

i. $X_{0,0}^r$ *and* $X_{0,0}^{r+1}$.

For d is even:

$$1+\frac{(q^{d-3}-1)(q^d+q^3-q^2-q)}{(q^3-1)(q^2-1)}-\frac{(q^d-q^3)(q^d+q^3)+q^d(q^d-q^6)}{(q^6-1)q^6}-(q+1)\left(\sum_{s=1}^{\frac{d-4}{2}} q^{2s}\frac{q^{6l(s)+6}-1}{q^6-1}\right)$$

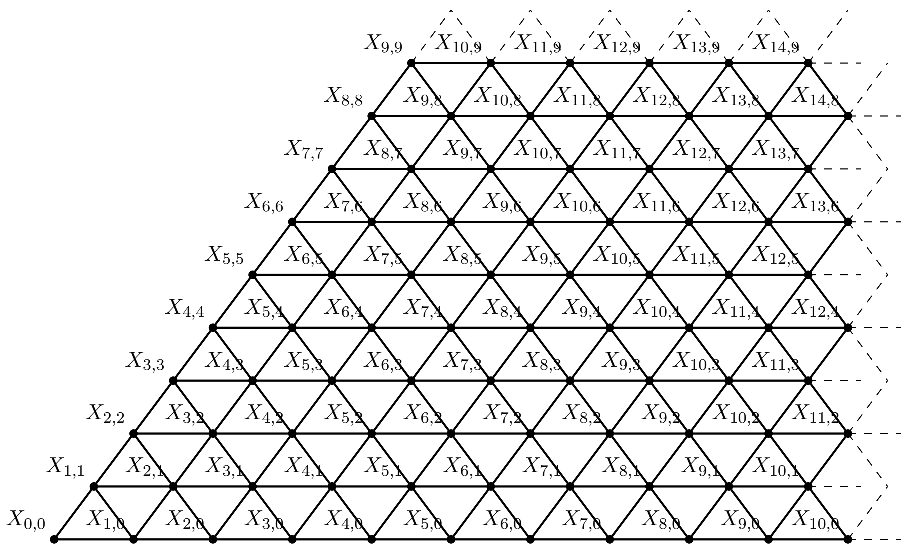
edges, if $d \geq 6$, between the vertices

i. $X_{0,0}^r$ *and* $X_{0,0}^{r+1}$.

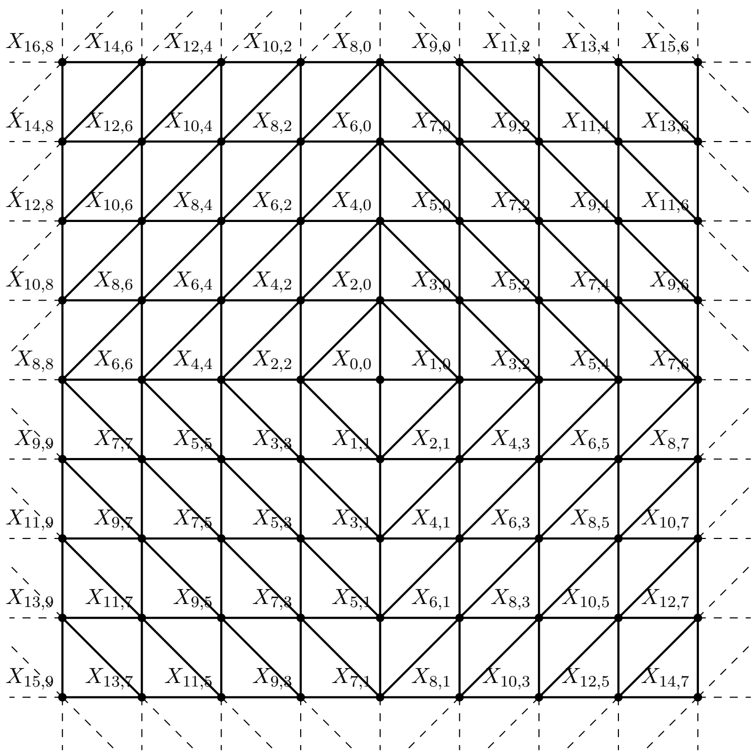
3.2. Examples

The quotient graphs for degrees $d \in \{1, 2, 3\}$:

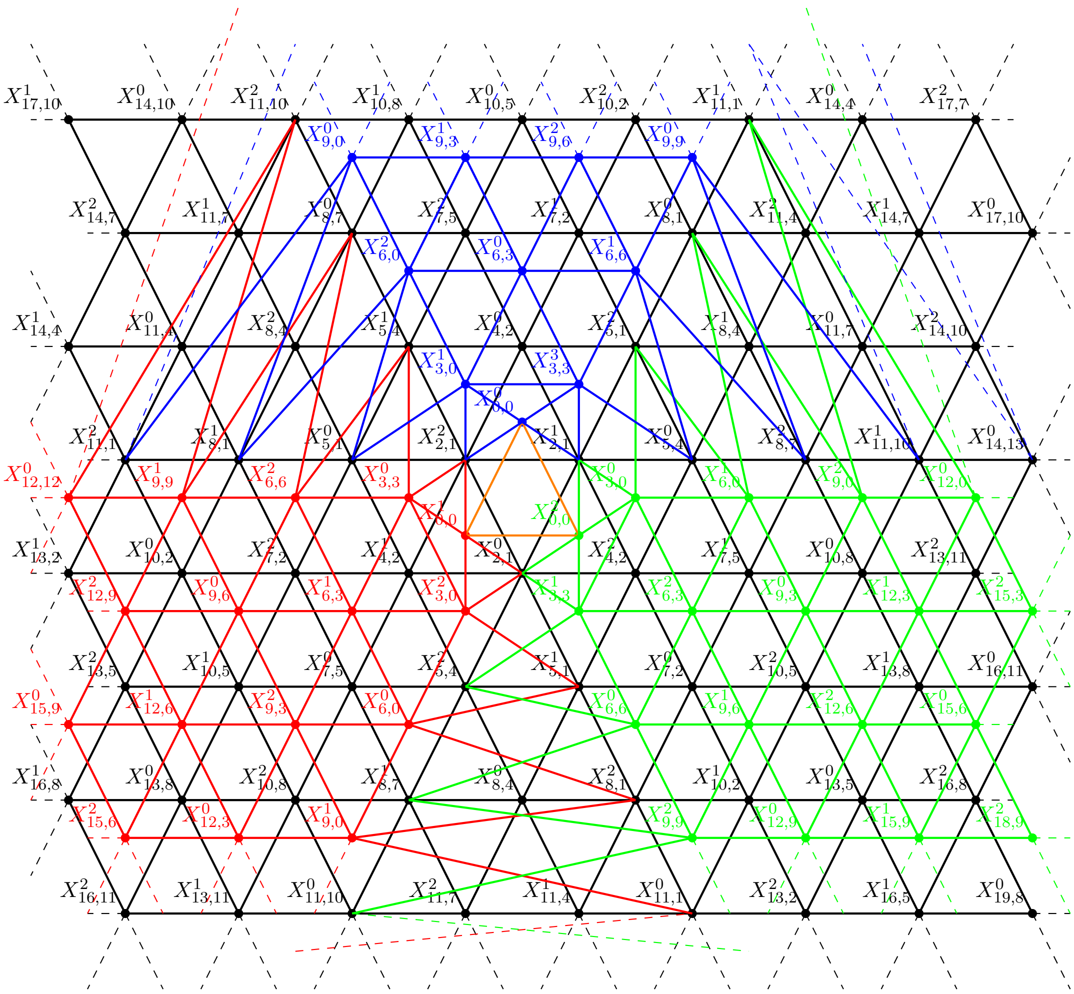
For degree 1:



For degree 2:



For degree 3:



3.3. Probability for polynomials to be coprime

Later, when we compute the cardinality of $\Upsilon_{n',m',n,m}$, we need the probability for arbitrary polynomials, where at least one of them is of degree greater zero, to be coprime. The following Lemma gives us this number:

Lemma 3.3.1 ([BB07], Corollary 5)**.** *Let $1 < m \in \mathbb{N}$, $d_1 \in \mathbb{N}$, $d_2, ..., d_m \in \mathbb{N}_0$ and for $1 \leq i \leq m$ let $a_i \in \mathbb{F}_q[t]$ be arbitrary polynomials of degree d_i, respectively. (Here we consider the zero polynomial as polynomial of degree 0). Then the probability that $a_1, ..., a_m$ are coprime is $1 - \frac{1}{q^{m-1}}$.*

Corollary 3.3.2. *Let $1 < m \in \mathbb{N}$, $d_1 \in \mathbb{N}$, $d_2, ..., d_m \in \mathbb{N}_0$ and let $a_1 \in \mathbb{F}_q[t]$ an arbitrary polynomial of degree d_1 and for $2 \leq i \leq m$ let $a_i \in \mathbb{F}_q[t]$ be arbitrary polynomials of degree at most d_i, respectively. (Here we consider the zero polynomial as polynomial of degree 0). Then the probability that $a_1, ..., a_m$ are coprime is $1 - \frac{1}{q^{m-1}}$.*

Proof. Follows immediately from Lemma 3.3.1. □

3.4. Stabilizers in $\mathrm{PGL}_3(\mathcal{O}_{\{p,\infty\}})$

Now we generalize the arguments in [KMS15].

Let k denote the finite field $\mathbb{F}_q$ with q elements and let p denote a place of degree d of the rational function field $k(t)$ such that the place p corresponds to an irreducible monic polynomial f and a valuation ν_p. Additionally we consider the place ∞ of $k(t)$, which is a place of degree 1 and corresponds to the valuation $\nu_\infty : k(t) \to \mathbb{Z} \cup \{\infty\}, \nu_\infty\left(\frac{a}{b}\right) = deg(b) - deg(a)$ for $a, b \in k[t]$ and $a \neq 0 \neq b$ and $\nu_\infty(0) = \infty$. Let X denote the underlying graph of the Bruhat-Tits building corresponding to the place p and Y be the underlying graph of the Bruhat-Tits building corresponding to the place ∞. Then the vertices of X and Y are given by lattice classes and the action of $\mathrm{PGL}_3(k(t))$ on X and Y is induced by the canonical action from the left of $\mathrm{PGL}_3(k(t))$ on the 3-dimensional lattices.

Definition 3.4.1. Let $\Pi := \mathrm{PGL}_3(\mathcal{O}_{\{p,\infty\}})$, $\Gamma := \mathrm{PGL}_3(\mathcal{O}_{\{p\}})$ and $\Xi := \mathrm{PGL}_3(\mathcal{O}_{\{\infty\}})$. With $\{e_1, e_2, e_3\}$ we denote the standard basis of the vector space $k(t)^3$. Define $x_{0,0}$ to be the vertex in X corresponding to the standard lattice $\mathcal{O}_p e_1 \oplus \mathcal{O}_p e_2 \oplus \mathcal{O}_p e_3$ and for all $n \geq m \geq 0$ set $y_{n,m} := [L_{n,m}]$ in Y (the lattices $L_{n,m}$ are defined in 2.6.5).

Remark 3.4.2. Let U be a dense subgroup of a topological group G which acts on a discrete set X. Then U and G have the same orbits.

Remark 3.4.3. Since $\mathcal{O}_{\{p,\infty\}}$ is a dense subring of $k(t)$ it follows that $\mathrm{SL}_3(\mathcal{O}_{\{p,\infty\}})$ is dense in $\mathrm{SL}_3(k(t))$. Because in the closure of $\mathrm{SL}_3(\mathcal{O}_{\{p,\infty\}})$ we have the matrices in $\mathrm{SL}_3(k(t))$ of

the form $\begin{pmatrix} \star & 0 & 0 \\ \star & \star & 0 \\ 0 & 0 & \star \end{pmatrix}$, $\begin{pmatrix} \star & 0 & 0 \\ 0 & \star & 0 \\ \star & 0 & \star \end{pmatrix}$, $\begin{pmatrix} \star & 0 & 0 \\ 0 & \star & 0 \\ 0 & \star & \star \end{pmatrix}$, $\begin{pmatrix} \star & 0 & 0 \\ 0 & \star & \star \\ 0 & 0 & \star \end{pmatrix}$, $\begin{pmatrix} \star & 0 & \star \\ 0 & \star & 0 \\ 0 & 0 & \star \end{pmatrix}$, $\begin{pmatrix} \star & \star & 0 \\ 0 & \star & 0 \\ 0 & 0 & \star \end{pmatrix}$.
These matrices generate $\mathrm{SL}_3(k(t))$.

Lemma 3.4.4. *The group* $\mathrm{PSL}_3(\mathcal{O}_{\{p,\infty\}})$ *acts with three orbits on either* X *or* Y*, i.e.* $\mathrm{PSL}_3(\mathcal{O}_{\{p,\infty\}})$ *acts type preservingly on the vertices of the underlying graphs* X *and* Y *of the Bruhat-Tits buildings and acts transitively on the set of vertices of the same type.*

Proof. By Lemma 2.5.8 the group $\mathrm{SL}_3(k(t))$ acts type preservingly and transitive on the set of vertices of the same type (in X or Y). According to Remark 3.4.2 and Remark 3.4.3 the subgroup $\mathrm{SL}_3(\mathcal{O}_{\{p,\infty\}})$ of $\mathrm{SL}_3(k(t))$ acts type preservingly and transitive on the set of vertices of the same type (in X or Y). This implies that the same is true for $\mathrm{PSL}_3(\mathcal{O}_{\{p,\infty\}})$. □

Lemma 3.4.5. *If* 3 *is not a divisor of the degree* d*, then* $\Pi = \mathrm{PGL}_3(\mathcal{O}_{\{p,\infty\}})$ *acts transitively on both* X *and* Y*. If* 3 *divides the degree* d*, then* Π *acts transitively on* X*, but type preservingly on the vertices of* Y*.*

Proof. Let M be a matrix in $\mathrm{GL}_3(\mathcal{O}_{\{p,\infty\}})$. Then $det(M)$ is an element of the set $\mathcal{O}^{\times}_{\{p,\infty\}} = \{\lambda \cdot f^n \mid \lambda \in k^{\times}, n \in \mathbb{Z}\}$. Thus $det(M) = \lambda \cdot f^n$ for some $\lambda \in k^{\times}$, $n \in \mathbb{Z}$. Hence $\nu_p(det(M)) = n$. Therefore Π acts non type preservingly and $\mathrm{PSL}_3(\mathcal{O}_{\{p,\infty\}}) \subset \Pi$ acts transitively on the set of vertices of the same type, which yields that Π acts transitively on X.

For the action on Y it is $\nu_\infty(det(M)) = -nd$, which is congruent to 0 (mod 3) if 3 divides d. In this case Π acts type preservingly on Y. If $d \equiv 1 \pmod 3$, then $\nu_\infty(det(M)) \equiv -n \pmod 3$ and hence Π acts non type preservingly on Y. Because $\mathrm{PSL}_3(\mathcal{O}_{\{p,\infty\}}) \subset \Pi$ acts transitively on vertices of Y of the same type, it follows, that Π acts transitively on Y. Similarly, if $d \equiv 2 \pmod 3$, then $\nu_\infty(det(M)) \equiv -2n \pmod 3$ and again Π acts transitively on Y. □

Proposition 3.4.6. *The stabilizer in* $\mathrm{PSL}_n(\mathcal{O}_{\{p,\infty\}})$ *of the vertex* $x_{0,0}$ *is* $\mathrm{PSL}_n(\mathcal{O}_{\{\infty\}})$ *and the stabilizer in* $\mathrm{PSL}_n(\mathcal{O}_{\{p,\infty\}})$ *of the vertex* $y_{0,0}$ *is* $\mathrm{PSL}_n(\mathcal{O}_{\{p\}})$*.*

Proof. With Corollary 2.5.11 it suffices to show

$$\mathrm{PSL}_n(\mathcal{O}_{\{p,\infty\}}) \cap \mathrm{PGL}_n(\mathcal{O}_\infty) = \mathrm{PSL}_n(\mathcal{O}_{\{p\}}) \text{ and } \mathrm{PSL}_n(\mathcal{O}_{\{p,\infty\}}) \cap \mathrm{PGL}_n(\mathcal{O}_p) = \mathrm{PSL}_n(\mathcal{O}_{\{\infty\}}):$$

Let $[g] \in \mathrm{PSL}_n(\mathcal{O}_{\{p,\infty\}}) \cap \mathrm{PGL}_n(\mathcal{O}_\infty)$ with $g = (g_{ij})_{1 \le i,j \le n} \in \mathrm{GL}_n(\mathcal{O}_\infty)$. Then there exist a $\lambda \in \mathcal{O}^{\times}_\infty$ such that $\lambda \cdot g \in \mathrm{SL}_n(\mathcal{O}_{\{p,\infty\}})$, this means $det(\lambda g) = \lambda^n det(g) = 1$. Then we have $\nu_\infty(\lambda) = 0$ and $\nu_\infty(\lambda \cdot g_{ij}) = \nu_\infty(g_{ij}) \ge 0$ for all $1 \le i,j \le n$. We conclude that $\lambda \cdot g$ is an element in $\mathrm{SL}_n(\mathcal{O}_\infty)$ and for all $1 \le i,j \le n$ the corresponding entry

$\lambda \cdot g_{ij}$ lies in the intersection $\mathcal{O}_{\{p,\infty\}} \cap \mathcal{O}_\infty = \mathcal{O}_{\{p\}}$. Therefore $\lambda \cdot g$ is in $\mathrm{SL}_n(\mathcal{O}_{\{p\}})$ and $[g] \in \mathrm{PSL}_n(\mathcal{O}_{\{p\}})$. For the other inclusion it is obvious that $\mathrm{PSL}_n(\mathcal{O}_{\{p\}})$ is a subset of $\mathrm{PSL}_n(\mathcal{O}_{\{p,\infty\}})$ and with $\mathcal{O}_{\{p\}} \subset \mathcal{O}_\infty$ it follows $\mathrm{PSL}_n(\mathcal{O}_{\{p\}}) \subset \mathrm{PGL}_n(\mathcal{O}_\infty)$.

The second equality follows by similar argumentation. □

Remark 3.4.7. For every ring R and natural number n the following sequence is exact: $1 \to \mathrm{PSL}_n(R) \to \mathrm{PGL}_n(R) \to R^\times/(R^\times)^n \to 1$, i.e. $R^\times/(R^\times)^n \cong \mathrm{PGL}_n(R)/\mathrm{PSL}_n(R)$.

Proposition 3.4.8. *a)* $\Pi_{x_{0,0}} = \Xi$.

b) $\Gamma_{x_{0,0}} = \mathrm{PGL}_3(k)$.

c) $\Pi_{y_{0,0}} = \tilde{\Gamma}$, *where in case* 3 *divides* d *the stabilizer* $\tilde{\Gamma}$ *contains* Γ *as an index* 3 *subgroup and in case* 3 *does not divides* d *the stabilizer* $\tilde{\Gamma}$ *equals* Γ.

d) $\Xi_{y_{0,0}} = \mathrm{PGL}_3(k)$.

Proof. a) First we compute the index of $\mathrm{PSL}_3(\mathcal{O}_{\{p,\infty\}})$ in $\mathrm{PGL}_3(\mathcal{O}_{\{p,\infty\}})$. To do this we want to use Remark 3.4.7. So we need the number of elements in $k^\times$ with a third root in $k^\times$. If $q \equiv 1 \pmod 3$, then $\frac{q-1}{3}$ elements in $k^\times$ have a third root in $k^\times$. If q is not congruent $1 \pmod 3$ every non-zero element in the field has a third root, i.e. there are $q-1$ elements in $k^\times$ having a third root in $k^\times$. Therefore

$[\mathrm{PGL}_3(\mathcal{O}_{\{p,\infty\}}) : \mathrm{PSL}_3(\mathcal{O}_{\{p,\infty\}})] = |\mathcal{O}^\times_{\{p,\infty\}}/(\mathcal{O}^\times_{\{p,\infty\}})^3|$

$$= |\{\lambda f^n \cdot (\mathcal{O}^\times_{\{p,\infty\}})^3 \mid \lambda \in k^\times, n \in \mathbb{Z}\}| = \begin{cases} \frac{q-1}{\frac{q-1}{3}} \cdot 3 = 9 & \text{if } q \equiv 1 \pmod 3, \\ \frac{q-1}{q-1} \cdot 3 = 3 & \text{else} \end{cases},$$

because $\lambda f^n \in (\mathcal{O}^\times_{\{p,\infty\}})^3$ if and only if there exists an integer m with $n = 3m$ and λ has a third root in $k^\times$.

Using again Remark 3.4.7 we compute $[\mathrm{PGL}_3(\mathcal{O}_{\{\infty\}}) : \mathrm{PSL}_3(\mathcal{O}_{\{\infty\}})] = |\mathcal{O}^\times_{\{\infty\}}/(\mathcal{O}^\times_{\{\infty\}})^3|$

$$= |k^\times/(k^\times)^3| = \begin{cases} 3 & \text{if } q \equiv 1 \pmod 3, \\ 1 & \text{else} \end{cases}.$$

By 3.4.5 and 3.4.4 the group $\mathrm{PGL}_3(\mathcal{O}_{\{p,\infty\}})$ acts transitive on X, while $\mathrm{PSL}_3(\mathcal{O}_{\{p,\infty\}})$ acts type preservingly on X, i.e. the orbit $\Pi(x_{0,0})$ is three times bigger than the orbit $\mathrm{PSL}_3(\mathcal{O}_{\{p,\infty\}})(x_{0,0})$. Because of Proposition 3.4.6 we know that $\mathrm{PSL}_3(\mathcal{O}_{\{\infty\}})$ is the stabilizer of $x_{0,0}$ in $\mathrm{PSL}_3(\mathcal{O}_{\{p,\infty\}})$.

Now the orbit-stabilizer theorem yields

$\mathrm{PSL}_3(\mathcal{O}_{\{p,\infty\}})/\mathrm{PSL}_3(\mathcal{O}_{\{\infty\}}) \cong \mathrm{PSL}_3(\mathcal{O}_{\{p,\infty\}})(x_{0,0})$ and $\Pi/\Pi_{x_{0,0}} \cong \Pi(x_{0,0})$. Since we know that $\Pi(x_{0,0})$ is three times bigger than

$\mathrm{PSL}_3(\mathcal{O}_{\{p,\infty\}})(x_{0,0})$ we conclude $\Pi/\Pi_{x_{0,0}}$ is three times bigger than

$\mathrm{PSL}_3(\mathcal{O}_{\{p,\infty\}})/\mathrm{PSL}_3(\mathcal{O}_{\{\infty\}})$, i.e. $[\Pi : \Pi_{x_{0,0}}] = 3\cdot[\mathrm{PSL}_3(\mathcal{O}_{\{p,\infty\}}) : \mathrm{PSL}_3(\mathcal{O}_{\{\infty\}})]$. With $[\Pi : \Pi_{x_{0,0}}][\Pi_{x_{0,0}} : \mathrm{PSL}_3(\mathcal{O}_{\{\infty\}})] = [\Pi : \mathrm{PSL}_3(\mathcal{O}_{\{\infty\}})]$

$= [\Pi : \mathrm{PSL}_3(\mathcal{O}_{\{p,\infty\}})][\mathrm{PSL}_3(\mathcal{O}_{\{p,\infty\}}) : \mathrm{PSL}_3(\mathcal{O}_{\{\infty\}})]$

it follows $3\cdot[\Pi_{x_{0,0}} : \mathrm{PSL}_3(\mathcal{O}_{\{\infty\}})] = [\Pi : \mathrm{PSL}_3(\mathcal{O}_{\{p,\infty\}})] = 3\cdot[\Xi : \mathrm{PSL}_3(\mathcal{O}_{\{\infty\}})]$, where the second equation holds because of the computations of the two indexes above. Thus $[\Pi_{x_{0,0}} : \mathrm{PSL}_3(\mathcal{O}_{\{\infty\}})] = [\Xi : \mathrm{PSL}_3(\mathcal{O}_{\{\infty\}})]$.

With Corollary 2.5.11 we get $\Xi \subseteq \mathrm{PGL}_3(\mathcal{O}_p) = \mathrm{PGL}_3(k(t))_{x_{0,0}}$ and with $\Xi \subseteq \Pi$ we obtain $\Xi \subseteq \Pi \cap \mathrm{PGL}_3(\mathcal{O}_p) = \Pi_{x_{0,0}}$, which yields $\Pi_{x_{0,0}} = \Xi$.

b) Due to 2.5.11 we have $\Gamma_{x_{0,0}} = \Gamma \cap \mathrm{PGL}_3(\mathcal{O}_p)$ and this intersection is equal to $\mathrm{PGL}_3(k)$, because let $[g] \in \Gamma \cap \mathrm{PGL}_3(\mathcal{O}_p)$ such that $g \in \mathrm{GL}_3(\mathcal{O}_p)$. Now we can find an element $\lambda \in \mathcal{O}_p^\times$ with $\lambda g \in \mathrm{GL}_3(\mathcal{O}_{\{p\}})$. So $\nu_p(\lambda) = 0$ and $det(\lambda g)$ is an element in $\mathcal{O}_{\{p\}}^\times = k^\times$. With $\nu_p(\lambda) = 0$ and $g \in \mathrm{GL}_3(\mathcal{O}_p)$ we deduce $\lambda g \in \mathrm{GL}_3(\mathcal{O}_p)$. This implies $\lambda g \in \mathrm{GL}_3(\mathcal{O}_p) \cap \mathrm{GL}_3(\mathcal{O}_{\{p\}}) = \mathrm{GL}_3(k)$. That $\mathrm{PGL}_3(k)$ is a subset of the intersection $\Gamma \cap \mathrm{PGL}_3(\mathcal{O}_p)$ follows since $k = \mathcal{O}_p \cap \mathcal{O}_{\{p\}}$.

c) If we consider the action of Π on Y we have to distinguish two cases, because of Lemma 3.4.5: In case 3 divides d the action of Π and of $\mathrm{PSL}_3(\mathcal{O}_{\{p,\infty\}})$ on Y are both type preserving and the orbits $\Pi(y_{0,0})$ and $\mathrm{PSL}_3(\mathcal{O}_{\{p,\infty\}})(y_{0,0})$ have the same length, while if 3 is not a divisor of d the orbit $\Pi(y_{0,0})$ is three times bigger than the orbit $\mathrm{PSL}_3(\mathcal{O}_{\{p,\infty\}})(y_{0,0})$. Similar to part a) we have

$[\Pi : \mathrm{PSL}_3(\mathcal{O}_{\{p,\infty\}})] = \begin{cases} \frac{q-1}{\frac{q-1}{3}} \cdot 3 = 9 & \text{if } q \equiv 1 \pmod 3, \\ \frac{q-1}{q-1} \cdot 3 = 3 & \text{else} \end{cases}$ and

$[\Gamma : \mathrm{PSL}_3(\mathcal{O}_{\{p\}})] = |\mathcal{O}_{\{p\}}^\times/(\mathcal{O}_{\{p\}}^\times)^3| = |k^\times/(k^\times)^3| = \begin{cases} 3 & \text{if } q \equiv 1 \pmod 3, \\ 1 & \text{else} \end{cases}$.

Together with $\mathrm{PGL}_3(\mathcal{O}_{\{p\}}) \subseteq \mathrm{PGL}_3(\mathcal{O}_\infty) = \mathrm{PGL}_3(k(t))_{y_{0,0}}$, here we use again 2.5.11, it follows $\Gamma \subseteq \Pi_{y_{0,0}}$. Thus, if 3 is a divisor of d the index of Γ in $\tilde{\Gamma}$ is $[\tilde{\Gamma} : \Gamma] = \frac{[\Pi:\mathrm{PSL}_3(\mathcal{O}_{\{p,\infty\}})]}{[\Gamma:\mathrm{PSL}_3(\mathcal{O}_{\{p\}})]} = 3$ and in the other cases $[\tilde{\Gamma} : \Gamma] = \frac{[\Pi:\mathrm{PSL}_3(\mathcal{O}_{\{p,\infty\}})]}{3[\Gamma:\mathrm{PSL}_3(\mathcal{O}_{\{p\}})]} = 1$, i.e. $\tilde{\Gamma} = \Gamma$.

d) Similar to part b): It is $\Xi_{y_{0,0}} = \mathrm{PGL}_3(\mathcal{O}_{\{\infty\}}) \cap \mathrm{PGL}_3(\mathcal{O}_\infty)$ by Corollary 2.5.11. Furthermore $\mathrm{PGL}_3(\mathcal{O}_{\{\infty\}}) \cap \mathrm{PGL}_3(\mathcal{O}_\infty) = \mathrm{PGL}_3(k)$, because let $[g] \in \mathrm{PGL}_3(\mathcal{O}_{\{\infty\}}) \cap \mathrm{PGL}_3(\mathcal{O}_\infty)$ with $g \in \mathrm{GL}_3(\mathcal{O}_\infty)$, then there exists a $\lambda \in \mathcal{O}_\infty^\times$ such that $\lambda g \in \mathrm{GL}_3(\mathcal{O}_{\{\infty\}})$, but this means $det(g) \in \mathcal{O}_{\{\infty\}}^\times = k^\times$ and it follows $\nu_\infty(\lambda) = 0$, i.e. $\lambda g \in \mathrm{GL}_3(\mathcal{O}_\infty) \cap \mathrm{GL}_3(\mathcal{O}_{\{\infty\}}) = \mathrm{GL}_3(k)$. Since $k = \mathcal{O}_\infty \cap \mathcal{O}_{\{\infty\}}$ we have $\mathrm{PGL}_3(\mathcal{O}_{\{\infty\}}) \cap \mathrm{PGL}_3(\mathcal{O}_\infty) \supseteq \mathrm{PGL}_3(k)$.

□

Remark 3.4.9. In part c) of the previous Proposition 3.4.8 we can take arbitrary n instead of 3. The proof stays the same. By Proposition 2.6.9 we know that $\Xi_{y_{n,m}} = H_{n,m}$ for all $n \geq m \geq 0$.

3.5. Maps sending $x_{0,0}$ to a neighbor and $y_{n,m}$ to $y_{n',m'}$

Here we still follow the strategy in [KMS15].

Lemma 3.5.1. *Let h be an element in $\mathrm{PGL}_3(\mathcal{O}_{\{p,\infty\}})$ represented by a matrix $M \in \mathrm{GL}_3(k[t])$, where not all entries of M are divisible by f as polynomials. Then the distance between $x_{0,0}$ and $h(x_{0,0})$ equals $\nu_p(det(M))$.*

Proof. Similarly to the proof of Lemma 3.1 in [KMS15]: Let L be a lattice in the lattice class $x_{0,0}$. Then we know by Remark 2.6.4 that the distance between $x_{0,0}$ and $h(x_{0,0})$ equals $b - a$, where b is minimal such that $f^b L$ is contained by ML and a is maximal such that $f^a L$ contains ML.

Due to the conditions on the entries of M, we have that L contains ML but fL does not, so $a = 0$.

Now we calculate b: We have that $f^b L \subseteq ML$ is equivalent to $L \supseteq f^b M^{-1} L$. The entries of M^{-1} are up to minus signs the first minors of M divided by $det(M) \in \mathcal{O}^{\times}_{\{p,\infty\}}$. Hence, if one wants to multiply M^{-1} with a power f^b of f such that in this product the entries lie in $k[t]$, then the minimal and sufficient such b is $\nu_p(det(M))$. $\square$

Proposition 3.5.2. *The set of elements of Π that map $x_{0,0}$ to a neighbor and $y_{n,m}$ to $y_{n',m'}$ for natural numbers $n \geq m \geq 0$ and $n' \geq m' \geq 0$ equals*

$$\Upsilon_{n',m',n,m} := \left\{ M := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix} \;\middle|\; \begin{array}{c} \alpha,\beta,\gamma,\delta,\varepsilon,\theta,\vartheta,\rho,\iota \in k[t], \\ deg(\alpha) \leq \frac{d-2n+2n'+m-m'}{3},\ deg(\beta) \leq \frac{d+n+2n'-2m-m'}{3}, \\ deg(\gamma) \leq \frac{d+n+2n'+m-m'}{3},\ deg(\delta) \leq \frac{d-2n-n'+m+2m'}{3}, \\ deg(\varepsilon) \leq \frac{d+n-n'-2m+2m'}{3},\ deg(\theta) \leq \frac{d+n-n'+m+2m'}{3}, \\ deg(\vartheta) \leq \frac{d-2n-n'+m-m'}{3},\ deg(\rho) \leq \frac{d+n-n'-2m-m'}{3}, \\ deg(\iota) \leq \frac{d+n-n'+m-m'}{3}, \\ det(M) = \lambda f, \lambda \in k^{\times} \end{array} \right\}$$

Proof. Consider an element $h \in \Pi$ with $h(x_{0,0})$ is adjacent to $x_{0,0}$. Then h is represented by a matrix

$$M' := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix} \in \mathrm{GL}_3(\mathcal{O}_{\{p,\infty\}})$$

and by multiplication with the appropriate power of f if necessary, we may assume that all the entries $\alpha, \beta, \gamma, \delta, \varepsilon, \theta, \vartheta, \rho$ and ι lie in $k[t]$ and are coprime as polynomials. Using Lemma 3.5.1 we know that the distance between $x_{0,0}$ and $h(x_{0,0})$ equals $\nu_p(det(M'))$. Since these two vertices are adjacent we have $\nu_p(det(M')) = 1$, in particular f divides $det(M') \in k[t]$, but f^2 does not divides the determinant of M'. With $det(M') \in \mathcal{O}^{\times}_{\{p,\infty\}}$ it follows the existence of a scalar $\lambda \in k^{\times}$ such that $det(M') = \lambda f$.

Next we want to determine the set of elements in $\mathrm{PGL}_3(k(t))$ which map $y_{n,m}$ to $y_{n',m'}$ for $n \geq m \geq 0$ and $n' \geq m' \geq 0$. By Proposition 2.5.11 the stabilizer of $y_{0,0}$ in $\mathrm{PGL}_3(k(t))$ is $\mathrm{PGL}_3(\mathcal{O}_\infty)$. Hence we can describe the set of elements mapping $y_{n,m}$ to $y_{n',m'}$ as $g' \mathrm{PGL}_3(\mathcal{O}_\infty) g$ with elements $g, g' \in \mathrm{PGL}_3(k(t))$ satisfying $g'(y_{0,0}) = y_{n',m'}$ and $g(y_{n,m}) = y_{0,0}$. In particular, g' can be represented by the matrix $\begin{pmatrix} t^{n'} & 0 & 0 \\ 0 & t^{m'} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} t^{-n} & 0 & 0 \\ 0 & t^{-m} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a possible representative for g. Now let

$$M'' := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix} := \begin{pmatrix} at^{n'-n} & bt^{n'-m} & ct^{n'} \\ dt^{m'-n} & et^{m'-m} & ft^{m'} \\ gt^{-n} & ht^{-m} & i \end{pmatrix} = \begin{pmatrix} t^{n'} & 0 & 0 \\ 0 & t^{m'} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} t^{-n} & 0 & 0 \\ 0 & t^{-m} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

be an arbitrary element in $g' \mathrm{PGL}_3(\mathcal{O}_\infty) g$. This implies that the elements

$$M'' = \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix} \in \mathrm{PGL}_3(k(t)),$$

which map $y_{n,m}$ to $y_{n',m'}$ are exactly those elements that satisfy

$$\alpha, \beta, \gamma, \delta, \varepsilon, \theta, \vartheta, \rho, \iota \in k(t),\ \nu_\infty(\alpha) \geq n - n',\ \nu_\infty(\beta) \geq m - n',\ \nu_\infty(\gamma) \geq -n',\ \nu_\infty(\delta) \geq n - m',\ \nu_\infty(\varepsilon) \geq m - m',\ \nu_\infty(\theta) \geq -m',\ \nu_\infty(\vartheta) \geq n,\ \nu_\infty(\rho) \geq m,\ \nu_\infty(\iota) \geq 0 \text{ and}$$

$$\nu_\infty(det(M'')) = \nu_\infty(t^{n'+m'}) + \nu_\infty\left(det\left(\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\right)\right) + \nu_\infty(t^{-n-m}) = n - n' + m - m'.$$

Next we describe the set $\Upsilon_{n',m',n,m}$ of elements in Π which map the vertex $x_{0,0}$ to a neighbor and $y_{n,m}$ to $y_{n',m'}$. To do so we want to use, as above, matrices

$$\tilde{M} := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix} \in \mathrm{GL}_3(\mathcal{O}_{\{p,\infty\}})$$

where the entries are coprime polynomials and the determinant of $\tilde{M}$ is a non-zero scalar multiple of f. Since $\nu_\infty(f) = -d$ we have to multiply the matrix $\begin{pmatrix} at^{n'-n} & bt^{n'-m} & ct^{n'} \\ dt^{m'-n} & et^{m'-m} & ft^{m'} \\ gt^{-n} & ht^{-m} & i \end{pmatrix}$ from above with the matrix $t^{\frac{d+n-n'+m-m'}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ to arrive at the desired set. $\square$

Remark 3.5.3. The set $\Upsilon_{n',m',n,m}$ is only in one of the following three cases non-empty: The first case is that $d \equiv 0 \pmod 3$ and $n + 2n' + m - m' \equiv 0 \pmod 3$, the second one is $d \equiv 1 \pmod 3$ and $n + 2n' + m - m' \equiv 2 \pmod 3$ and in the third case we have $d \equiv 2 \pmod 3$ and $n + 2n' + m - m' \equiv 1 \pmod 3$. This is because all the degree restraints of the entries of a matrix in $\Upsilon_{n',m',n,m}$ are congruent $\frac{d+n+2n'+m-m'}{3} \pmod 3$, what implies that $d + n + 2n' + m - m'$ has to be divisible by 3, since otherwise we can not have that the determinant of a matrix in $\Upsilon_{n',m',n,m}$ is a non-zero scalar multiple of the polynomial f.

Remark 3.5.4. If we set $\kappa := \frac{d-2n-n'+m-m'}{3}$ then we have the following description of the above set in 3.5.2:

$$\Upsilon_{n',m',n,m} = \left\{ M := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix} \;\middle|\; \begin{array}{c} \alpha, \beta, \gamma, \delta, \varepsilon, \theta, \vartheta, \rho, \iota \in k[t], \\ deg(\alpha) \le \kappa + n',\ deg(\beta) \le \kappa + n + n' - m, \\ deg(\gamma) \le \kappa + n + n',\ deg(\delta) \le \kappa + m', \\ deg(\varepsilon) \le \kappa + n - m + m',\ deg(\theta) \le \kappa + n + m', \\ deg(\vartheta) \le \kappa,\ deg(\rho) \le \kappa + n - m, \\ deg(\iota) \le \kappa + n, \\ det(M) = \lambda f, \lambda \in k^\times \end{array} \right\}.$$

Furthermore, with 3.5.3 we have $\Upsilon_{n',m',n,m} \neq \emptyset$ if and only if $\kappa \in \mathbb{Z}$.

Remark 3.5.5. The set $\Upsilon_{n',m',n,m}$ is stable under multiplication with matrices in $H_{n',m'}$ from the left and similarly, $\Upsilon_{n',m',n,m}$ is stable under multiplication with elements in $H_{n,m}$ from the right.

Remark 3.5.6. If we have $\kappa > 0$ then it is not possible to have two entries in the same row or the same column of M equal to zero. This holds because the determinant of M is then a product of two polynomials that have both less degree than the degree d of the irreducible polynomial f. Hence we have a contradiction. For instance, if $\vartheta = 0 = \rho$, then

$$\lambda f = det(M) = \iota(\alpha\varepsilon - \delta\beta),$$

which is a contradiction to the Irreducibility of f, since

$$deg(\iota) \leq \kappa + n < \kappa + n + 2\kappa + n + n' - m + m' = d \text{ and}$$
$$deg(\alpha\varepsilon - \delta\beta) \leq 2\kappa + n + n' - m + m' < 2\kappa + n + n' - m + m' + \kappa + n = d.$$

Similarly, we get a contradiction in the case where two entries in the same row or column of M are equal to zero.

Lemma 3.5.7. *For natural numbers $n \geq m \geq 0$ and $n' \geq m' \geq 0$ let $\kappa = \frac{d-2n-n'+m-m'}{3} > 0$ be a natural number and take $M := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix} \in \Upsilon_{n',m',n,m}$.*

a)
- *For polynomials $a, b \in k[t]$ with $deg(a) \leq n'$ and $deg(b) \leq n'$ the equations $a(\delta, \varepsilon) + b(\vartheta, \rho) = 0$ or $a(\theta, \delta) + b(\iota, \vartheta) = 0$ or $a(\theta, \varepsilon) + b(\iota, \rho) = 0$ imply $a = 0 = b$.*
- *For polynomials $a, b \in k[t]$ with $deg(a) \leq n$ and $deg(b) \leq n$ the equations $a(\vartheta, \delta) + b(\rho, \varepsilon) = 0$ or $a(\alpha, \vartheta) + b(\beta, \rho) = 0$ or $a(\alpha, \delta) + b(\beta, \varepsilon) = 0$ imply $a = 0 = b$.*
- *For polynomials $a, b \in k[t]$ with $deg(a) \leq 0$ and $deg(b) \leq n'$ the equation $a(\alpha, \beta) + b(\vartheta, \rho) = 0$ implies $a = 0 = b$.*
- *For polynomials $a, b \in k[t]$ with $deg(a) \leq n$ and $deg(b) \leq 0$ the equation $a(\vartheta, \delta) + b(\iota, \theta) = 0$ implies $a = 0 = b$.*

b) For $a \in k$, $b, c \in k[t]$ with $deg(b) \leq n'$ and $deg(c) \leq n'$ the equation $a(\alpha, \beta) + b(\delta, \varepsilon) + c(\vartheta, \rho) = 0$ implies $a = 0 = b = c$.

Proof. a) In order to show $a = 0 = b$ we assume that a or b has to be non-zero. But if we assume one of them to be zero and the other one is non-zero, then we find in all cases two entries in the same row or the same column of M equal to zero. Due to Remark 3.5.6 this is not possible in the case $\kappa > 0$. Therefore we can assume that a and b are both non-zero polynomials. Notice that in all cases we know that the corresponding first minor of M has to be zero.

Now we consider the first case $deg(a) \leq n'$ and $deg(b) \leq n'$.

Consider first $a(\delta, \varepsilon) + b(\vartheta, \rho) = 0$ and $\delta\rho - \vartheta\varepsilon = 0$. It follows

$$\lambda f = det(M) = \alpha(\varepsilon\iota - \rho\theta) - \beta(\delta\iota - \vartheta\theta) =$$
$$\alpha(\varepsilon\iota - (-b^{-1}a\varepsilon)\theta) - \beta(\delta\iota - (-b^{-1}a\delta)\theta) = (\alpha\varepsilon - \delta\beta)(\iota + b^{-1}a\theta).$$

Since f is irreducible we have a contradiction, because $deg(\alpha\varepsilon - \delta\beta) \leq 2\kappa + n + n' - m + m' = d - \kappa - n < d$ and $deg(\iota + b^{-1}a\theta) \leq \kappa + n + m' + n' = d - 2\kappa - n + m < d$, since $deg(b^{-1}a) \leq deg(a) \leq n'$. Similarly, we do the other cases:

With $a(\theta, \delta) + b(\iota, \vartheta) = 0$ we deduce

$$\lambda f = det(M) = \alpha(\varepsilon\iota - \rho\theta) + \gamma(\delta\rho - \vartheta\varepsilon) =$$
$$\alpha(\varepsilon(-b^{-1}a\theta) - \rho\theta) + \gamma(\delta\rho - (-b^{-1}a\delta)\varepsilon) = (\gamma\delta - \alpha\theta)(\rho + b^{-1}a\varepsilon).$$

Since f is irreducible we have a contradiction, because $deg(\gamma\delta - \alpha\theta) \leq 2\kappa + n + n' + m' = d - \kappa - n + m < d$ and $deg(\rho + b^{-1}a\varepsilon) \leq \kappa + n - m + m' + n' = d - 2\kappa - n < d$.

For $a(\theta, \varepsilon) + b(\iota, \rho) = 0$ we obtain

$$\lambda f = det(M) = \gamma(\delta\rho - \vartheta\varepsilon) - \beta(\delta\iota - \vartheta\theta) = (\beta\theta - \gamma\varepsilon)(\vartheta + b^{-1}a\delta).$$

We have a contradiction to the Irreducibility of f, because $deg(\beta\theta - \gamma\varepsilon) \leq d - \kappa < d$ and $deg(\vartheta + b^{-1}a\delta) \leq \kappa + m' + n' = d - 2\kappa - n + m - n < d$.

Now we suppose $deg(a) \leq n$ and $deg(b) \leq n$.

In the case $a(\vartheta, \delta) + b(\rho, \varepsilon) = 0$ the determinant is given by

$$\lambda f = det(M) = \alpha(\varepsilon\iota - \rho\theta) - \beta(\delta\iota - \vartheta\theta) = \alpha(-b^{-1}a)(\delta\iota - \vartheta\theta) - \beta(\delta\iota - \vartheta\theta) =$$
$$(\vartheta\theta - \delta\iota)(\beta + b^{-1}a\alpha).$$

With the Irreducibility of f we obtain a contradiction, because $deg(\vartheta\theta - \delta\iota) \leq d - \kappa - n + m - n' < d$ and $deg(\iota + b^{-1}a\theta) \leq \kappa + n + n' = d - 2\kappa - n + m - m' < d$, since $deg(b^{-1}a) \leq deg(a) \leq n$.

Using $a(\alpha, \vartheta) + b(\beta, \rho) = 0$ we get

$$\lambda f = det(M) = \iota(\alpha\varepsilon - \delta\beta) + \gamma(\delta\rho - \vartheta\varepsilon) = (\alpha\iota - \gamma\vartheta)(\varepsilon + b^{-1}a\delta).$$

Since f is irreducible we have a contradiction, because $deg(\alpha\iota - \gamma\vartheta) \leq d - \kappa - n + m - m' < d$ and $deg(\varepsilon + b^{-1}a\delta) \leq \kappa + n + m' = d - 2\kappa - n + m - n' < d$.

From $a(\alpha, \delta) + b(\beta, \varepsilon) = 0$ we obtain

$$\lambda f = det(M) = \vartheta(\beta\theta - \varepsilon\gamma) - \rho(\alpha\theta - \delta\gamma) = (\delta\gamma - \alpha\theta)(\rho + b^{-1}a\vartheta).$$

This is a contradiction to the Irreducibility of f, because $deg(\delta\gamma - \alpha\theta) \leq d - \kappa - n + m < d$ and $deg(\rho + b^{-1}a\vartheta) \leq \kappa + n = d - 2\kappa - n + m - n' - m' < d$.

Next we do the case $deg(a) \leq 0$ and $deg(b) \leq n'$. Due to $a(\alpha, \beta) + b(\vartheta, \rho) = 0$ the determinant is given by

$$\lambda f = det(M) = \iota(\alpha\varepsilon - \delta\beta) + \gamma(\delta\rho - \vartheta\varepsilon) = (\alpha\varepsilon - \delta\beta)(\iota + b^{-1}a\gamma).$$

Since f is irreducible we have a contradiction, because $deg(\alpha\varepsilon - \delta\beta) \leq 2\kappa + n + n' - m + m' = d - \kappa - n < d$ and $deg(\iota + b^{-1}a\gamma) \leq \kappa + n + n' = d - 2\kappa - n + m - m' < d$, since $deg(b^{-1}a) \leq deg(a) \leq 0$.

The last case is $deg(a) \leq n$ and $deg(b) \leq 0$. According to $a(\vartheta, \delta) + b(\iota, \theta) = 0$ we know

$$\lambda f = det(M) = \alpha(\varepsilon\iota - \rho\theta) + \gamma(\delta\rho - \vartheta\varepsilon) = (\delta\rho - \vartheta\varepsilon)(\gamma + b^{-1}a\alpha).$$

Due to the fact that f is irreducible we have a contradiction, because $deg(\delta\rho - \vartheta\varepsilon) \leq d - \kappa - n - n' < d$ and $deg(\gamma + b^{-1}a\alpha) \leq \kappa + n + n' = d - 2\kappa - n + m - m' < d$, since $deg(b^{-1}a) \leq deg(a) \leq n$.

b) Let $a(\alpha,\beta)+b(\delta,\varepsilon)+c(\vartheta,\rho)=0$ for some $a\in k$, $b,c\in k[t]$ with $deg(b)\le n'$ and $deg(c)\le n'$. Now assume $a\in k^\times$. As we saw in Remark 3.5.6 we can not have two entries of the same row or column of M equal to zero. Hence $(\alpha,\beta)\neq(0,0)$. According to $a(\alpha,\beta)+b(\delta,\varepsilon)+c(\vartheta,\rho)=0$ we deduce

$$\alpha\varepsilon-\delta\beta=(-a^{-1}(b\delta+c\vartheta))\varepsilon-\delta(-a^{-1}(b\varepsilon+c\rho))=-a^{-1}c(\vartheta\varepsilon-\delta\rho)\text{ and}$$
$$\alpha\rho-\vartheta\beta=(-a^{-1}(b\delta+c\vartheta))\rho-\vartheta(-a^{-1}(b\varepsilon+c\rho))=-a^{-1}b(\delta\rho-\vartheta\varepsilon).$$

Therefore

$$\lambda f=det(M)=\iota(\alpha\varepsilon-\delta\beta)-\theta(\alpha\rho-\vartheta\beta)+\gamma(\delta\rho-\vartheta\varepsilon)=$$
$$\iota(-a^{-1}c(\vartheta\varepsilon-\delta\rho))-\theta(-a^{-1}b(\delta\rho-\vartheta\varepsilon))+\gamma(\delta\rho-\vartheta\varepsilon)=(a^{-1}c\iota+a^{-1}b\theta+\gamma)(\delta\rho-\vartheta\varepsilon).$$

This is a contradiction, since f is irreducible, $deg(a^{-1}c\iota+a^{-1}b\theta+\gamma)\le\kappa+n+m'+n'=d-2\kappa-n+m<d$ and $deg(\delta\rho-\vartheta\varepsilon)\le d-\kappa-n-n'<d$. We conclude that a has to be zero. But then the remaining equation $b(\delta,\varepsilon)+c(\vartheta,\rho)=0$ implies $b=0=c$, because of part a) of the Lemma.

□

3.5.1. The cardinality of $\Upsilon_{n',m',n,m}$

In the next part we want to determine the cardinality of $\Upsilon_{n',m',n,m}$. In some cases we can calculate the size of $\Upsilon_{n',m',n,m}$ by using the results of [KMS15], because we have some 2×2 submatrix with the same properties as in [KMS15]. This is done in Lemma 3.5.9. If we can not use [KMS15] to compute the cardinality of $\Upsilon_{n',m',n,m}$, we have to count the elements in this set in another way. We do this in Lemma 3.5.12 and Lemma 3.5.13.

Remark 3.5.8. To compute the cardinality of the set $\Upsilon_{n',m',n,m}$ we have the following two different ways. We can calculate all possible choices for a matrix $M=\begin{pmatrix}\alpha&\beta&\gamma\\ \delta&\varepsilon&\theta\\ \vartheta&\rho&\iota\end{pmatrix}\in\Upsilon_{n',m',n,m}$ in order to count all the elements in $\Upsilon_{n',m',n,m}$, but we could also use the matrix $M^\tau:=\begin{pmatrix}\iota&\theta&\gamma\\ \rho&\varepsilon&\beta\\ \vartheta&\delta&\alpha\end{pmatrix}$ instead of M to count the elements in $\Upsilon_{n',m',n,m}$, because there is a bijection between $\Upsilon_{n',m',n,m}$ and the set $\{M^\tau\mid M\in\Upsilon_{n',m',n,m}\}$. Note that $det(M)=det(M^\tau)$. If we consider M^τ instead of M, for the degree restraints of the entries, the roles of n and n' are interchanged and the roles of $n-m$ and m' are interchanged. But since $\kappa=\frac{d-2n-n'+m-m'}{3}=\frac{d-n-n'-(n-m)-m'}{3}$, we see that κ stays the same if we fix the degree d and interchange the roles of n with n' and $n-m$ with m'. So we get

$$|\Upsilon_{n',m',n,m}|=|\Upsilon^\tau_{n,n-m,n',n'-m'}|.$$

Lemma 3.5.9. *Let $\kappa < 0$, $\Upsilon_{n',m',n,m} \neq \emptyset$ and $n \geq m \geq 0$ and $n' \geq m' \geq 0$ be natural numbers.*

1. *If $\kappa + m' < 0 = \kappa + n'$, this implies that all matrices $M \in \Upsilon_{n',m',n,m}$ are of the form $M = \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \varepsilon & \theta \\ 0 & \rho & \iota \end{pmatrix}$, then the cardinality of the set $\Upsilon_{n',m',n,m}$ is given by*
$$|\Upsilon_{n',m',n,m}| = (q-1)^2(q^d + q^{d-1})(q^2 - q)q^{deg(\beta)+deg(\gamma)+2},$$
where $deg(\beta)$ (respective $deg(\gamma)$) denotes the maximal possible degree for β (respective γ).

2. *If $\kappa + n - m < 0 = \kappa + n$, i.e. all matrices $M \in \Upsilon_{n',m',n,m}$ are of the form $M = \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ 0 & 0 & \iota \end{pmatrix}$, then the cardinality of the set $\Upsilon_{n',m',n,m}$ is given by*
$$|\Upsilon_{n',m',n,m}| = (q-1)^2(q^d + q^{d-1})(q^2 - q)q^{deg(\theta)+deg(\gamma)+2},$$
where $deg(\theta)$ (respective $deg(\gamma)$) denotes the maximal possible degree for θ (respective γ).

Proof. 1. Suppose $\kappa + m' < 0 = \kappa + n'$, then we know that for all matrices $M = \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix} \in \Upsilon_{n',m',n,m}$ the entries δ and ϑ are always equal to zero and we have $det(M) = \alpha(\varepsilon\iota - \rho\theta) = \lambda f$, for some non-zero scalar λ. Notice that $deg(\alpha) = \kappa + n' = 0$. Since α is a non-zero element in the field k there is a $\mu \in k^\times$ with $\varepsilon\iota - \rho\theta = \mu f$. Now we are in the same situation like in [KMS15] if we want to count the possibilities for the matrix $\begin{pmatrix} \varepsilon & \theta \\ \rho & \iota \end{pmatrix}$. Hence we can use the same arguments as in [KMS15] to find $(q-1)^2(q^d + q^{d-1})(q^2 - q)$ possibilities for this kind of matrices (in [KMS15] the given number is divided by $q - 1$, because it is considered in the projective group; we will divide by $q - 1$ later). Next we have to count the possible choices for the first row of M. Since α is a non-zero element in k, we have $q - 1$ choices for α. Moreover, we have $q^{deg(\beta)+deg(\gamma)+2}$ possibilities for the last two entries β and γ. Because we are working in the projective group, we divide by $q - 1$. Therefore we obtain
$$|\Upsilon_{n',m',n,m}| = (q-1)^2(q^d + q^{d-1})(q^2 - q)q^{deg(\beta)+deg(\gamma)+2}$$
in this case.

2. Due to 3.5.8 we can compute the possibilities for the matrix M^τ instead of the possibilities for M. According to the first part, which we proved above, we find

$$|\Upsilon_{n',m',n,m}| = (q-1)^2(q^d + q^{d-1})(q^2-q)q^{deg(\theta)+deg(\gamma)+2}.$$

□

Definition 3.5.10. Define $\Upsilon^{\vartheta=0}_{n',m',n,m}$ to be the subset of $\Upsilon_{n',m',n,m}$ consisting of all matrices, where the lower left entry ϑ is always 0. Similar we define $\Upsilon^{\vartheta\neq 0}_{n',m',n,m}$ to be the subset of $\Upsilon_{n',m',n,m}$ of all matrices with lower left entry ϑ different from 0. Let $\kappa = \frac{d-2n-n'+m-m'}{3}$ and $u := 2\kappa + n' + n - m + m'$.

Remark 3.5.11. When we have $\kappa + m' \geq 0$ and $\kappa + n - m \geq 0$, which means the degree restraints for the entries of a matrix in $\Upsilon_{n',m',n,m}$ are such that it is not the case that two entries in a row or a column necessarily have to be zero, then we can not apply Lemma 3.5.9. To calculate the cardinality of the set $\Upsilon_{n',m',n,m}$ in this case we state the following two Lemmata:

Lemma 3.5.12. *Let $\Upsilon_{n',m',n,m} \neq \emptyset$. If $\kappa + m' \geq 0$ and $\kappa + n - m \geq 0$, then for the cardinality of $\Upsilon_{n',m',n,m}$ the following holds:*

1. *For $\kappa < 0$:*
   a) *Is $-\kappa < m'$, then $|\Upsilon_{n',m',n,m}| = (q-1)^2(q^2-1)^2 q^{d+2\kappa+2n-m+n'+m'+1}$*
   b) *Is $-\kappa = m' < n'$ and $\kappa + n - m > 0$,*
   then $|\Upsilon_{n',m',n,m}| = (q-1)^2(q^2-1)q^{d+\kappa+2n+n'-m+3}$
   c) *Is $-\kappa = m' = n'$ and $\kappa + n - m > 0$,*
   then $|\Upsilon_{n',m',n,m}| = (q-1)(q^2-1)^2 q^{d+2n-m+2}$
2. *For $\kappa = 0$:*
   a) *Is $0 < m'$, then $|\Upsilon^{\vartheta=0}_{n',m',n,m}| = (q-1)(q^2-1)q^{n'+m'-1}(q-1)(q^2-1)q^{d+2n-m+2}$*
   and $|\Upsilon^{\vartheta\neq 0}_{n',m',n,m}| = (q-1)q^{n'+m'+2}(q-1)(q^2-1)q^{d+2n-m+2}$,
   in particular $|\Upsilon_{n',m',n,m}| = (q-1)^2(q^2-1)(q^3+q^2-1)q^{d+2n-m+n'+m'+1}$
   b) *Is $0 = m' < n'$ and $n > m$,*
   then $|\Upsilon^{\vartheta=0}_{n',m',n,m}| = (q-1)q^{n'+1}(q-1)(q^2-1)q^{d+2n-m+2}$
   and $|\Upsilon^{\vartheta\neq 0}_{n',m',n,m}| = (q-1)q^{n'+2}(q-1)(q^2-1)q^{d+2n-m+2}$,
   in particular $|\Upsilon_{n',m',n,m}| = (q-1)(q^2-1)^2 q^{d+2n-m+n'+3}$
   c) *Is $0 = m' = n'$ and $n > m$,*
   then $|\Upsilon^{\vartheta=0}_{n',m',n,m}| = (q^2-1)(q-1)(q^2-1)q^{d+2n-m+2}$
   and $|\Upsilon^{\vartheta\neq 0}_{n',m',n,m}| = (q-1)q^2(q-1)(q^2-1)q^{d+2n-m+2}$,
   in particular $|\Upsilon_{n',m',n,m}| = (q^3-1)(q-1)(q^2-1)q^{d+2n-m+2}$

3. *For $\kappa > 0$ it is $|\Upsilon_{n',m',n,m}| = (q-1)^2(q^2-1)^2(q^2+q+1)q^{2d}$.*

Proof. Let $\Upsilon_{n',m',n,m} \neq \emptyset$ and $M = \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix} \in \Upsilon_{n',m',n,m}$. We want to determine the cardinality of $\Upsilon_{n',m',n,m}$ by computing all possible choices of the matrix M:

From the definition of $\Upsilon_{n',m',n,m}$ we deduce that

$$deg(\alpha) \leq \kappa + n', \quad deg(\beta) \leq \kappa + n + n' - m, \quad deg(\gamma) \leq \kappa + n + n', \quad deg(\delta) \leq \kappa + m', \quad deg(\varepsilon) \leq \kappa + n - m + m', \quad deg(\theta) \leq \kappa + n + m', \quad deg(\vartheta) \leq \kappa, \quad deg(\rho) \leq \kappa + n - m, \quad deg(\iota) \leq \kappa + n.$$

Suppose $\kappa + m'$ and $\kappa + n - m$ are both greater or equal to zero. Furthermore, if we have $\kappa + m' = 0$, we assume $\kappa + n - m > 0$. Notice that these assumptions imply $\kappa + u > 0$.

1. First we choose the first column of the matrix $M \in \Upsilon_{n',m',n,m}$ in such a way that $\alpha, \delta, \vartheta$ are coprime polynomials and at least one of these three polynomials has its maximal possible degree.

   In order to choose the first column with these properties, we have to consider three different cases: First let $\kappa < 0$, which means $\vartheta = 0$ and we have to choose α and δ coprime and such that one of these two polynomials has its maximal possible degree. To do this, we have to consider again three different cases, namely $-\kappa < m'$, $-\kappa = m' < n'$ and $-\kappa = m' = n'$.

   According to 3.3.2 we know that the probability of two arbitrary chosen polynomials, where at least one of them has degree greater than zero, to be coprime is $\frac{q-1}{q}$.

   In the case $-\kappa < m'$, we have

   $$\tfrac{q-1}{q}((q-1)q^{2\kappa+n'+m'+1} + (q-1)q^{2\kappa+n'+m'}) = (q-1)(q^2-1)q^{2\kappa+n'+m'-1}$$

   possible choices for the first column of M.

   If $-\kappa = m' < n'$, then $deg(\delta) \leq 0$, which implies that $\alpha \in k[t]$ and $\delta \in k$ are coprime polynomials if and only if δ is non-zero. Therefore, we have in this case

   $$(q-1)q^{2\kappa+n'+m'+1} = (q-1)q^{\kappa+n'+1}$$

   possibilities.

   For $-\kappa = m' = n'$ it follows $deg(\alpha) \leq 0$ and $deg(\delta) \leq 0$. Since α and δ should be coprime, we have to choose $(\alpha, \delta) \in k^2 \setminus \{(0,0)\}$. This yields (q^2-1) possible choices.

   Next we consider the case $\kappa = 0$, which means $deg(\vartheta) \leq 0$. Now there are two possibilities: $\vartheta = 0$ or $\vartheta \neq 0$. Let $\vartheta = 0$, then we have the same three cases as in the case $\kappa < 0$, but replacing $-\kappa$ by 0, i.e. the three cases are $0 < m'$, $0 = m' < n'$ and $0 = m' = n'$. The number of possible choices in these three cases is given by the corresponding solution in the case $\kappa < 0$ if we replace κ by 0.

If $\vartheta \neq 0$, it follows that $\alpha, \delta, \vartheta$ are automatically coprime. This implies

$$(q-1)q^{n'+m'+2}$$

possibilities for the first column of M, where $\vartheta \neq 0$.

To get the number of all possible choices of the first column of M in the case $\kappa = 0$, we add the choices for $\vartheta = 0$ and $\vartheta \neq 0$.

The last case is $\kappa > 0$. By 3.3.2, the probability of three arbitrary chosen polynomials, where at least one of them has degree greater than zero, to be coprime is given by $\frac{q^2-1}{q^2}$. Hence, there are

$$\frac{q^2-1}{q^2}((q-1)^3 q^{d-2n+m} + 3(q-1)^2 q^{d-2n+m} + 3(q-1)q^{d-2n+m})$$
$$= (q^2-1)(q-1)q^{d-2n+m-2}(q^2+q+1)$$

possibilities for the first column of the matrix M.

2. Choose the second column of M, such that the entries β, ε, ρ are coprime polynomials and such that additionally the polynomials $\alpha\varepsilon - \delta\beta$, $\alpha\rho - \vartheta\beta$, $\delta\rho - \vartheta\varepsilon$ are coprime and at least one of these three polynomials has its maximal possible degree.

   a) First we choose polynomials a, b, c of degree $deg(a) \leq 2\kappa + n' + n - m + m' = u$, $deg(b) \leq 2\kappa + n' + n - m = u - m'$ and $deg(c) \leq 2\kappa + n - m + m' = u - n'$, such that $a = \alpha\varepsilon - \delta\beta$, $b = \alpha\rho - \vartheta\beta$, $c = \delta\rho - \vartheta\varepsilon$.

      Notice that $u - n' \geq 0$, since in the case $\kappa < 0$ we assume $-\kappa \leq m'$ and $-\kappa \leq n - m$. If we want to write a, b, c in such a way, then the polynomials a, b, c have to fulfill the equation $a\vartheta - b\delta + c\alpha = 0$. Consider the set of solutions of the following system of linear equations: $Ax = 0$, where

$$A = \begin{pmatrix}
\alpha_0 & 0 & & 0 & -\delta_0 & 0 & & 0 & \vartheta_0 & 0 & & 0 \\
\vdots & \alpha_0 & \ddots & & \vdots & -\delta_0 & \ddots & & \vdots & \vartheta_0 & \ddots & \\
\alpha_{\kappa+n'} & \vdots & & & -\delta_{\kappa+m'} & \vdots & & & \vartheta_\kappa & \vdots & & \\
0 & \alpha_{\kappa+n'} & \ddots & 0 & 0 & -\delta_{\kappa+m'} & \ddots & 0 & 0 & \vartheta_\kappa & \ddots & 0 \\
\vdots & 0 & \ddots & \alpha_0 & \vdots & 0 & \ddots & -\delta_0 & \vdots & 0 & \ddots & \vartheta_0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \alpha_{\kappa+n'} & 0 & 0 & 0 & -\delta_{\kappa+m'} & 0 & 0 & 0 & \vartheta_\kappa
\end{pmatrix}$$

$$\in k^{(\kappa+u+1)\times(3u-n'-m'+3)} \text{ and } x = \begin{pmatrix} c_0 \\ \vdots \\ c_{u-n'} \\ b_0 \\ \vdots \\ b_{u-m'} \\ a_0 \\ \vdots \\ a_u \end{pmatrix}.$$

W.l.o.g. let α be of maximal possible degree, i.e. of degree $\kappa+n'$ and moreover, $\delta = 0$ is only possible if $\vartheta = 0$.

We want to show, that the matrix A has maximal rank. Therefore, consider the submatrix B, where we take the first $u - n' + 1$ columns of the columns with entries α, the first $\kappa + n' - deg(gcd(\alpha, \delta))$ columns of the columns with entries $-\delta$ and $deg(gcd(\alpha, \delta))$ columns of the columns with entries ϑ. We show that the resulting matrix $B \in k^{(\kappa+u+1)\times(\kappa+u+1)}$ has maximal rank, i.e. $rank(B) = \kappa + u + 1$.

For the proof, we consider three cases:

First case: For $deg(gcd(\alpha, \delta)) = 0$, i.e. the polynomials α and δ are coprime, there are two cases possible.

If $\delta = 0$, then we assumed ϑ to be zero, too. Since the three polynomials $\alpha, \delta, \vartheta$ are coprime we deduce $\alpha \in k^{\times}$. Furthermore, $deg(\alpha) = \kappa + n'$ yields $\kappa + n' = 0$. Hence $B = \begin{pmatrix} \alpha & & \\ & \ddots & \\ & & \alpha \end{pmatrix} \in k^{(\kappa+u+1)\times(\kappa+u+1)}$ has obviously rank $\kappa + u + 1$.

Now if α and δ are both non-zero, then the matrix B looks like

$$B = \begin{pmatrix}
\alpha_0 & 0 & & & 0 & -\delta_0 & 0 & & 0 \\
\vdots & \alpha_0 & \ddots & & & \vdots & -\delta_0 & \ddots & \\
\alpha_{\kappa+n'} & \vdots & \ddots & & \vdots & -\delta_{\kappa+m'} & \vdots & \ddots & 0 \\
0 & \alpha_{\kappa+n'} & \ddots & & & 0 & -\delta_{\kappa+m'} & \ddots & -\delta_0 \\
\vdots & 0 & \ddots & & & \vdots & 0 & \ddots & \vdots \\
 & & 0 & \ddots & 0 & \vdots & \vdots & & -\delta_{\kappa+m'} \\
 & & & \ddots & \alpha_0 & 0 & & & 0 \\
\vdots & & & \ddots & \vdots & \vdots & & \ddots & \vdots \\
0 & & & 0 & \alpha_{\kappa+n'} & 0 & & & 0
\end{pmatrix}$$

$\in k^{(\kappa+u+1)\times(\kappa+u+1)}$.

Here the last block of 0 under the columns where the entries are the coefficients of $-\delta$ consists of $\kappa + u + 1 - (2\kappa + n' + m' + 1) = \kappa + n - m \geq 0$ rows.

Consider a linear combination of the columns of B equal to zero. This means there exist polynomials g and h such that $deg(g) \leq u - n'$, $deg(h) \leq \kappa + n' - 1$ and $g\alpha - h\delta = 0$, hence $g\alpha = h\delta$. Since α and δ are coprime it follows α divides h. But $deg(\alpha) = \kappa + n'$ and $deg(h) \leq \kappa + n' - 1$, whence $h = 0$ and $g = 0$. We deduce that the columns of B are linearly independent and $rank(B) = \kappa + u + 1$.

Second case: If $deg(gcd(\alpha, \delta)) = \kappa + n' > 0$, in particular we have $gcd(\alpha, \delta) = \alpha$ and note that in this case we necessarily have $n' = m'$, because the assumption $\delta = 0$ implies $\alpha \in k^\times$ and hence $\kappa + n' = deg(\alpha) = 0$. Moreover, α and ϑ are coprime, since α, δ and ϑ are coprime polynomials. Then

$$B = \begin{pmatrix}
\alpha_0 & 0 & & & 0 & \vartheta_0 & 0 & & 0 \\
\vdots & \alpha_0 & \ddots & & & \vdots & \vartheta_0 & \ddots & \\
\alpha_{\kappa+n'} & \vdots & \ddots & & \vdots & \vartheta_\kappa & \vdots & \ddots & 0 \\
0 & \alpha_{\kappa+n'} & \ddots & & & 0 & \vartheta_\kappa & \ddots & \vartheta_0 \\
\vdots & 0 & \ddots & & & \vdots & 0 & \ddots & \vdots \\
 & & 0 & \ddots & 0 & \vdots & \vdots & & \vartheta_\kappa \\
 & & & \ddots & \alpha_0 & 0 & & & 0 \\
\vdots & & & \ddots & \vdots & \vdots & & \ddots & \vdots \\
0 & & & 0 & \alpha_{\kappa+n'} & 0 & & & 0
\end{pmatrix}$$

$\in k^{(\kappa+u+1)\times(\kappa+u+1)}$ and it follows similar to the first case that $rank(B) =$

$\kappa + u + 1$. Note that in this case the 0 block in the lower right corner of B has $\kappa + u + 1 - (2\kappa + n' + 1) = \kappa + n - m + m' = \kappa + n - m + n' > 0$ rows.

Third case: Now let $0 < deg(gcd(\alpha, \delta)) < \kappa + n'$ and consider a linear combination of the columns of B equal to zero. Therefore, there exist polynomials g, h and p, such that

$$deg(g) \leq u - n',\ deg(h) \leq \kappa + n' - deg(gcd(\alpha, \delta)) - 1,\ deg(p) \leq deg(gcd(\alpha, \delta)) - 1 \text{ and } \alpha g - \delta h + \vartheta p = 0.$$

This implies $gcd(\alpha, \delta)$ divides ϑp and since α, δ and ϑ are coprime, we know that $gcd(\alpha, \delta)$ divides p. But $deg(p) \leq deg(gcd(\alpha, \delta)) - 1$, whence $p = 0$. So we have $\alpha g = \delta h$. From this equation we deduce $\frac{\alpha}{gcd(\alpha,\delta)}$ divides h. But then

$$deg(\tfrac{\alpha}{gcd(\alpha,\delta)}) = \kappa + n' - deg(gcd(\alpha, \delta)) \text{ and } deg(h) \leq \kappa + n' - deg(gcd(\alpha, \delta)) - 1$$

implies $h = 0$, whence $g = 0$. Therefore, $rank(B) = \kappa + u + 1$.

Since B is a submatrix of A, it follows that the rank of A is $\kappa + u + 1$, too. In particular, the matrix A has maximal rank. For the equation $Ax = 0$ we have

$$q^{3u-n'-m'+3-rank(A)} = q^{3u-n'-m'+3-(\kappa+u+1)} = q^{2u-\kappa-n'-m'+2} = q^{d-m+2}$$

solutions.

b) To fulfill the condition that at least one of the three polynomials a, b, c has its maximal possible degree we subtract all the solutions of the equation $\tilde{A}\tilde{x} = 0$, where $\tilde{A}$ is the matrix arising from A if we omit the last row of A and the $u - n' + 1$ column, the $2u - n' - m' + 2$ column and the last column of A and the variable $\tilde{x}$ has three rows less than x. In particular, we erase the $u - n' + 1$, the $2u - n' - m' + 2$ and the last row of x to obtain $\tilde{x}$. Now the matrix $\tilde{A}$ has rank $\kappa + u$, i.e. maximal rank. In order to see this, we consider the submatrix $\tilde{B} \in k^{(\kappa+u)\times(\kappa+u)}$ of $\tilde{A}$ arising from B if we omit the last row of B and the $u - n' + 1$ column of B. Since the last row of B has only one entry different from zero, in particular the only non-zero entry in the last row of B is the $u - n' + 1$ entry, we can use similar arguments like in the proof of $rank(B)$ is maximal to deduce that the rank of $\tilde{B}$ is also maximal, i.e. $rank(\tilde{B}) = \kappa + u$. This means that the matrix $\tilde{A}$ has maximal rank, in particular, $rank(\tilde{A}) = \kappa + u$. Hence we have

$$q^{3u-n'-m'-rank(\tilde{A})} = q^{3u-n'-m'-(\kappa+u)} = q^{2u-\kappa-n'-m'} = q^{d-m}$$

solutions for the equation $\tilde{A}\tilde{x} = 0$. If we subtract these solutions from the solutions of the equation $Ax = 0$, we have

$$q^{d-m+2} - q^{d-m} = q^{d-m}(q^2 - 1)$$

solutions left. All these solutions fulfill that at least one of the three polynomials a, b, c has its maximal possible degree.

c) Next, we ensure that the three polynomials a, b and c are coprime, which means we multiply with the probability for a, b, c to be coprime. Since $a = \alpha\varepsilon - \delta\beta, \quad b = \alpha\rho - \vartheta\beta, \quad c = \delta\rho - \vartheta\varepsilon$ are not independent polynomials we have to compute the probability of these certain three polynomials to be coprime.

Claim: The probability of a, b, c to be coprime is equal to $1 - \frac{1}{q}$.

Proof: Let $A, x, \tilde{A}, \tilde{x}$ as before. Moreover, define
$\mathbb{L} := \left\{ \left(g\frac{\tilde{a}}{\tilde{g}}, g\frac{\tilde{b}}{\tilde{g}}, g\frac{\tilde{c}}{\tilde{g}} \right) \in k[t]^3 \mid \begin{array}{c} (\tilde{a}, \tilde{b}, \tilde{c}) \text{ is a solution of } \tilde{A}\tilde{x} = 0, \tilde{g} := gcd(\tilde{a}, \tilde{b}, \tilde{c}) \\ \text{and } g \in k[t] \text{ with } deg(g) = deg(\tilde{g}) + 1 \end{array} \right\}$
and let $\mathbb{L}_{NC}$ be the set of all not coprime solutions of the system of linear equations $Ax = 0$.

Then $\mathbb{L} = \mathbb{L}_{\mathbb{NC}}$.

" $\subseteq$ " If we have some element $z := \left(g\frac{\tilde{a}}{\tilde{g}}, g\frac{\tilde{b}}{\tilde{g}}, g\frac{\tilde{c}}{\tilde{g}} \right) \in \mathbb{L}$ then the entries are not coprime, since $g = gcd\left(g\frac{\tilde{a}}{\tilde{g}}, g\frac{\tilde{b}}{\tilde{g}}, g\frac{\tilde{c}}{\tilde{g}} \right)$ and $deg(g) = deg(\tilde{g}) + 1 \geq 1$.

Furthermore, z is a solution of the equation $Ax = 0$, because $\alpha g\frac{\tilde{c}}{\tilde{g}} + \vartheta g\frac{\tilde{a}}{\tilde{g}} = \frac{g}{\tilde{g}}(\alpha\tilde{c} + \vartheta\tilde{a}) = \frac{g}{\tilde{g}}\delta\tilde{b} = \delta g\frac{\tilde{b}}{\tilde{g}}$, since $(\tilde{a}, \tilde{b}, \tilde{c})$ is a solution of $\tilde{A}\tilde{x} = 0$.

" $\supseteq$ " Take $(a, b, c) \in \mathbb{L}_{NC}$ and let $g := gcd(a, b, c)$. Then $deg(g) \geq 1$. It follows $(a, b, c) = (g\underline{a}, g\underline{b}, g\underline{c})$, with $\underline{a}, \underline{b}, \underline{c}$ are coprime and $\alpha\underline{c} + \vartheta\underline{a} = \delta\underline{b}$, since (a, b, c) is a solution of $Ax = 0$. Now choose some $\tilde{g} \in k[t]$ with $deg(\tilde{g}) = deg(g) - 1 \geq 0$ and define $\tilde{a} = \underline{a}\tilde{g}, \quad \tilde{b} = \underline{b}\tilde{g}, \quad \tilde{c} = \underline{c}\tilde{g}$. They fulfill the equation $\alpha\tilde{c} + \vartheta\tilde{a} = \alpha\underline{c}\tilde{g} + \vartheta\underline{a}\tilde{g} = \tilde{g}(\alpha\underline{c} + \vartheta\underline{a}) = \tilde{g}\delta\underline{b} = \delta\tilde{b}$. Hence $(a, b, c) = (g\underline{a}, g\underline{b}, g\underline{c}) = \left(g\frac{\tilde{a}}{\tilde{g}}, g\frac{\tilde{a}}{\tilde{g}}, g\frac{\tilde{a}}{\tilde{g}} \right) \in \mathbb{L}$.

Therefore we have

$$|\mathbb{L}_{NC}| = |\mathbb{L}| = q^{d-m}\frac{q^{deg(g)+1}}{q^{deg(\tilde{g})+1}} = q^{d-m+1}.$$

Let $\mathbb{S}$ be the set of solutions of $Ax = 0$. From the previous computations we know that $|\mathbb{S}| = q^{d-m+2}$. Then the probability for a solution of $Ax = 0$ to be not coprime is given by

$$\frac{|\mathbb{L}_{NC}|}{|\mathbb{S}|} = \frac{q^{d-m+1}}{q^{d-m+2}} = \frac{1}{q}.$$

This yields that the probability for a coprime solution of $Ax = 0$ is equal to $1 - \frac{1}{q}$.

Whence we have

$$q^{d-m}(q^2-1)\left(1-\frac{1}{q}\right) = q^{d-m-1}(q^2-1)(q-1)$$

possibilities for a, b, c with the required properties.

d) Since the three polynomials a, b, c are coprime and at least one of them has its maximal possible degree it follows that β, ε, ρ are coprime and at least one of these three polynomials has its maximal possible degree.

The equations $a = \alpha\varepsilon - \delta\beta, \quad b = \alpha\rho - \vartheta\beta, \quad c = \delta\rho - \vartheta\varepsilon$ imply that one of the polynomials a, b, c has its maximal possible degree if and only if the relevant polynomials from $\alpha, \delta, \vartheta$ and β, ε, ρ are of their respective maximal possible degree. Moreover, if we take $g = gcd(\beta, \varepsilon, \rho)$, then we have that g divides a, b, c, whence we find g divides $gcd(a, b, c) = 1$. Therefore $g = 1$ and hence β, ε, ρ are coprime.

e) For our chosen a, b, c we compute all possibilities for β, ε, ρ, such that $a = \alpha\varepsilon - \delta\beta, \quad b = \alpha\rho - \vartheta\beta, \quad c = \delta\rho - \vartheta\varepsilon$. Therefore we solve the following system of equations: $Ax = b$, where $x = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{\kappa+n+n'-m} \\ \varepsilon_0 \\ \vdots \\ \varepsilon_{\kappa+n-m+m'} \\ \rho_0 \\ \vdots \\ \rho_{\kappa+n-m} \end{pmatrix}$, $b = \begin{pmatrix} a_0 \\ \vdots \\ a_u \\ -b_0 \\ \vdots \\ -b_{u-m'} \\ c_0 \\ \vdots \\ c_{u-n'} \end{pmatrix}$ and

$$A = \left(\begin{array}{cccccccccccc}
-\delta_0 & 0 & & & \alpha_0 & 0 & & & & & & \\
\vdots & -\delta_0 & & & \vdots & \alpha_0 & & & & & & \\
-\delta_{\kappa+m'} & \vdots & \ddots & & & \vdots & \ddots & & & & & \\
 & -\delta_{\kappa+m'} & \ddots & & \alpha_{\kappa+n'} & & \ddots & & & & & \\
 & & \ddots & -\delta_0 & & \alpha_{\kappa+n'} & \ddots & \alpha_0 & & & & \\
 & & & \vdots & & & & \vdots & & & & \\
 & & & -\delta_{\kappa+m'} & & & & \alpha_{\kappa+n'} & & & & \\
\vartheta_0 & 0 & & & & & & & -\alpha_0 & 0 & & \\
\vdots & \vartheta_0 & & & & & & & \vdots & -\alpha_0 & & \\
\vartheta_\kappa & \vdots & \ddots & & & & & & -\alpha_{\kappa+n'} & \vdots & \ddots & \\
 & \vartheta_\kappa & \ddots & & & & & & & -\alpha_{\kappa+n'} & \ddots & \\
 & & \ddots & \vartheta_0 & & & & & & & \ddots & -\alpha_0 \\
 & & & \vdots & & & & & & & & \vdots \\
 & & & \vartheta_\kappa & & & & & & & & -\alpha_{\kappa+n'} \\
 & & & & -\vartheta_0 & 0 & & & \delta_0 & 0 & & \\
 & & & & \vdots & -\vartheta_0 & & & \vdots & \delta_0 & & \\
 & & & & -\vartheta_\kappa & \vdots & \ddots & & \delta_{\kappa+m'} & \vdots & \ddots & \\
 & & & & & -\vartheta_\kappa & \ddots & & & \delta_{\kappa+m'} & \ddots & \\
 & & & & & & \ddots & -\vartheta_0 & & & \ddots & \delta_0 \\
 & & & & & & & \vdots & & & & \vdots \\
 & & & & & & & -\vartheta_\kappa & & & & \delta_{\kappa+m'}
\end{array} \right)$$

$$\in k^{(3u-n'-m'+3)\times(3\kappa+3n+n'-3m+m'+3)} = K^{(2d-n-m+3)\times(d+n-2m+3)}$$

We claim $rank(A) = d - m + 2$.

Consider a linear combination of scalar multiples of the columns of A equal to zero. To have such a combination is equivalent to have polynomials g, h, p with

$$g(-\delta) + h\alpha = 0, \quad g\vartheta - p\alpha = 0, \quad -h\vartheta + p\delta = 0.$$

These equations are obviously fulfilled for $g = \alpha, \quad h = \delta, \quad p = \vartheta$. So we have to erase at least $n-m+1$ columns in order to get one of the polynomials g, h, p of lower degree as the corresponding polynomial $\alpha, \delta, \vartheta$, because otherwise we can choose $g = \alpha, \quad h = \delta, \quad p = \vartheta$ as a nontrivial linear combination equal to zero. Thus erase $n - m + 1$ columns, such that

$$deg(g) \leq \kappa + n + n' - m - (n - m + 1) = \kappa + n' - 1 = deg(\alpha) - 1.$$

According to $g\delta = h\alpha$ we deduce $\frac{\alpha}{gcd(\alpha,\delta)}$ divides g and $g\vartheta = p\alpha$ yields $\frac{\alpha}{gcd(\alpha,\vartheta)}$ divides g. From these two conditions it follows that α divides g, since α, δ and ϑ are coprime. Because of the degree restraints this implies $g = 0$. Using the above equations we derive $p = 0 = h$. Hence

$$rank(A) = d + n - 2m + 3 - (n - m + 1) = d - m + 2$$

as required.

Now we arrive at $q^{d+n-2m+3-rank(A)} = q^{d+n-2m+3-(d-m+2)} = q^{n-m+1}$ solutions for the system of linear equations $Ax = 0$, i.e. we have

$$q^{n-m+1}$$

possible choices for the polynomials β, ε, ρ.

For the second column of M we have

$$q^{d-m-1}(q^2 - 1)(q - 1)q^{n-m+1} = (q^2 - 1)(q - 1)q^{d+n-2m}$$

possible choices.

3. The determinant of M is given by λf for some non-zero element λ in the field k. For this element λ we have $q - 1$ possible choices.

4. Next we want to compute the possibilities for the last column of our matrix M. As before, we have chosen the polynomials $a = \alpha\varepsilon - \delta\beta, \quad b = \alpha\rho - \vartheta\beta, \quad c = \delta\rho - \vartheta\varepsilon$ with $deg(a) \leq 2\kappa + n' + n - m + m' = u$, $deg(b) \leq 2\kappa + n' + n - m = u - m'$ and $deg(c) \leq 2\kappa + n - m + m' = u - n'$. Then we can compute the last column of M by solving the following system of linear equations: $Ax = b$, where $x = \begin{pmatrix} \iota_0 \\ \vdots \\ \iota_{\kappa+n} \\ \theta_0 \\ \vdots \\ \theta_{\kappa+n+m'} \\ \gamma_0 \\ \vdots \\ \gamma_{\kappa+n+n'} \end{pmatrix}$,

$$b = \lambda \cdot \begin{pmatrix} f_0 \\ \vdots \\ f_d \end{pmatrix} \text{ and}$$

$$A = \begin{pmatrix}
a_0 & 0 & & & -b_0 & 0 & & & c_0 & 0 & & \\
\vdots & a_0 & & & \vdots & -b_0 & & & \vdots & c_0 & & \\
a_u & \vdots & \ddots & & -b_{u-m'} & \vdots & \ddots & & c_{u-n'} & \vdots & \ddots & \\
 & a_u & \ddots & a_0 & & -b_{u-m'} & \ddots & -b_0 & & c_{u-n'} & \ddots & c_0 \\
 & & \ddots & \vdots & & & \ddots & \vdots & & & \ddots & \vdots \\
 & & & a_u & & & & -b_{u-m'} & & & & c_{u-n'}
\end{pmatrix}$$

$\in k^{(d+1)\times(3\kappa+3n+n'+m'+3)} = k^{(d+1)\times(d+n+m+3)}$

In order to solve this system of linear equations we show $rank(A) = d+1$ is maximal. Let

$$B = \begin{pmatrix}
a_0 & 0 & & & & -b_0 & 0 & & & & c_0 & 0 & & \\
\vdots & a_0 & & & & \vdots & -b_0 & & & & \vdots & c_0 & & \\
a_u & \vdots & \ddots & & & -b_{u-m'} & \vdots & \ddots & & & c_{u-n'} & \vdots & \ddots & \\
 & a_u & \ddots & & & & -b_{u-m'} & \ddots & & & & c_{u-n'} & \ddots & c_0 \\
 & & \ddots & & & & & \ddots & & & & & \ddots & \vdots \\
 & & & \ddots & & & & & \ddots & & & & & c_{u-n'} \\
 & & & \ddots & a_0 & & & & \ddots & -b_0 & 0 & & & 0 \\
 & & & \ddots & \vdots & & & & \ddots & \vdots & \vdots & & & \vdots \\
 & & & & a_u & & & & & -b_{u-m'} & 0 & & & 0
\end{pmatrix}$$

$\in k^{(d-m)\times(d-m)}$.

So B is the submatrix of A, which we get if we erase the columns $\kappa+n-m+1$ until $\kappa+n+1$, $2\kappa+2n+m'-m+2$ until $2\kappa+2n+m'+2$ and the last $n+1$ columns of A. In total we erase $2m+2+n+1 = 2m+n+3$ columns of A. The lower right block of 0 entries in B consists of $\kappa+u+n-m+1-(u-n'+1+\kappa+n') = n-m \geq 0$ rows. Moreover, from $\kappa+u > 0$ we know $d-n = \kappa+u > 0$ and whence $d-m \geq d-n > 0$

Then we claim $rank(B) = d-m$ is maximal.

Consider a linear combination of the columns of B equal to zero. This means we have polynomials g, h, p with

$$deg(g) \le \kappa + n - m - 1, \quad deg(h) \le \kappa + n + m' - m - 1, \quad deg(p) \le \kappa + n' - 1$$
$$\text{and } ag - bh + cp = 0.$$

Notice that the degree restraint for h is greater or equal to 0 and additionally at least one of the degree restraints for g and p are also greater or equal to 0. This follows from our assumtion that $\kappa + m' > 0$ or $\kappa + n - m > 0$.

We have $0 = ag - bh + cp = det\left(\begin{pmatrix} \alpha & \beta & p \\ \delta & \varepsilon & h \\ \vartheta & \rho & g \end{pmatrix}\right)$, which means that the columns of the matrix $\begin{pmatrix} \alpha & \beta & p \\ \delta & \varepsilon & h \\ \vartheta & \rho & g \end{pmatrix} \in k(t)^{3\times 3}$ are linearly dependent. This implies that there exist $\mu_1, \mu_2, \mu_3 \in k(t)$ not all of them equal to zero, such that

$$\mu_1 \begin{pmatrix} \alpha \\ \delta \\ \vartheta \end{pmatrix} + \mu_2 \begin{pmatrix} \beta \\ \varepsilon \\ \rho \end{pmatrix} + \mu_3 \begin{pmatrix} p \\ h \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Actually we have $\mu_3 \neq 0$, because the first two columns of M are linearly independent.

If $\mu_2 = 0$, then there exists a $\nu = \frac{\nu_1}{\nu_2} \in K(t)$ with $\nu_1, \nu_2 \in k[t]$ and $\nu \begin{pmatrix} \alpha \\ \delta \\ \vartheta \end{pmatrix} = \begin{pmatrix} p \\ h \\ g \end{pmatrix}$.
Since $\alpha, \delta, \vartheta$ are coprime and p, h, g are polynomials it follows that $deg(\nu_2) = 0$. According to $deg(p) < \kappa + n' = deg(\alpha)$ it is not possible to have $deg(\nu_1) \geq 0$, i.e. we have $\nu = 0$ and $p = 0 = h = g$.

Assume $\mu_2 \neq 0$, then there exist $\nu, \mu \in k(t)$ with $\nu \begin{pmatrix} \alpha \\ \delta \\ \vartheta \end{pmatrix} + \mu \begin{pmatrix} p \\ h \\ g \end{pmatrix} = \begin{pmatrix} \beta \\ \varepsilon \\ \rho \end{pmatrix}$. This implies

$$a = \alpha\varepsilon - \delta\beta = \alpha(\delta\nu + \mu h) - \delta(\nu\alpha + \mu p) = \mu(\alpha h - \delta p), \quad b = \alpha\rho - \vartheta\beta =$$
$$\mu(\alpha g - \vartheta p), \quad c = \delta\rho - \vartheta\varepsilon = \mu(\delta g - \vartheta h).$$

Since a, b, c are coprime we deduce $\nu_\infty(\mu) \geq 0$. Moreover,

$$deg(\alpha h) \le \kappa + n' + \kappa + n - m + m' - 1 = u - 1 < u, \quad deg(\delta p) \le \kappa + m' + \kappa + n' - 1 = u - n + m - 1 < u, \quad deg(\alpha g) \le \kappa + n' + \kappa + n - m - 1 = u - m' - 1 < u - m', \quad deg(\vartheta p) \le \kappa + \kappa + n' - 1 = u - m' + m - n - 1 < u - m', \quad deg(\delta g) \le \kappa + m' + \kappa + n - m - 1 = u - n' - 1 < u - n', \quad deg(\vartheta h) \le \kappa + \kappa + n - m + m' - 1 = u - n' - 1 < u - n'$$

means that the three polynomials a, b, c have not maximal possible degree, a contradiction to the condition that at least one of the three polynomials a, b, c has maximal possible degree. This proves $rank(B) = d - m$.

Next consider the submatrix B' of A consisting of all columns in A that also occur in the matrix B and $m+1$ columns where the entries are the coefficients of the polynomial from a, b, c which has maximal possible degree and such that the leading coefficient is in the $d-m+1$ until the $d+1$ row. Since B has maximal rank it follows that B' has maximal rank. So B' is a submatrix of A with maximal possible rank, hence A has maximal rank, i.e. $rank(A) = d+1$.

Therefore, for the last column of M we have

$$q^{d+n+m+3-rank(A)} = q^{n+m+2}$$

possibilities.

Now we have to calculate the product of the possibilities for the columns of M (they are computed in 1, 2 and 4) and the choices for λ (cf. 3) in order to arrive at the stated cardinality in every case. Remember that we have to divide by $q-1$ because we are working in the projective group.

□

Lemma 3.5.13. *Let $\Upsilon_{n',m',n,m} \neq \emptyset$. If $\kappa + m' \geq 0$ and $\kappa + n - m \geq 0$, then for the cardinality of $\Upsilon_{n',m',n,m}$ the following holds:*

1. *For $\kappa < 0$:*
   a) *Is $-\kappa < n-m$, then $|\Upsilon_{n',m',n,m}| = (q-1)^2(q^2-1)^2 q^{d+2\kappa+2n-m+n'+m'+1}$*
   b) *Is $-\kappa = n-m < n$ and $\kappa + m' > 0$,*
      then $|\Upsilon_{n',m',n,m}| = (q-1)^2(q^2-1)q^{d+\kappa+n+n'+m'+3}$
   c) *Is $-\kappa = n-m = n$ and $\kappa + m' > 0$,*
      then $|\Upsilon_{n',m',n,m}| = (q-1)(q^2-1)^2 q^{d+n'+m'+2}$
2. *For $\kappa = 0$:*
   a) *Is $0 < n-m$, then $|\Upsilon^{\vartheta=0}_{n',m',n,m}| = (q-1)(q^2-1)q^{2n-m-1}(q-1)(q^2-1)q^{d+n'+m'+2}$*
      and $|\Upsilon^{\vartheta\neq0}_{n',m',n,m}| = (q-1)q^{2n-m+2}(q-1)(q^2-1)q^{d+n'+m'+2}$,
      in particular $|\Upsilon_{n',m',n,m}| = (q-1)^2(q^2-1)(q^3+q^2-1)q^{d+2n-m+n'+m'+1}$
   b) *Is $0 = n-m < n$ and $m' > 0$,*
      then $|\Upsilon^{\vartheta=0}_{n',m',n,m}| = (q-1)q^{n+1}(q-1)(q^2-1)q^{d+n'+m'+2}$
      and $|\Upsilon^{\vartheta\neq0}_{n',m',n,m}| = (q-1)q^{n+2}(q-1)(q^2-1)q^{d+n'+m'+2}$,
      in particular $|\Upsilon_{n',m',n,m}| = (q-1)(q^2-1)^2 q^{d+n+m'+n'+3}$

c) Is $0 = n - m = n$*, i.e.* $m = 0 = n$*, and additionally* $m' > 0$*,*
then $|\Upsilon^{\vartheta=0}_{n',m',n,m}| = (q^2-1)(q-1)(q^2-1)q^{d+n'+m'+2}$
and $|\Upsilon^{\vartheta\neq 0}_{n',m',n,m}| = (q-1)q^2(q-1)(q^2-1)q^{d+n'+m'+2}$*,*
in particular $|\Upsilon_{n',m',n,m}| = (q^3-1)(q-1)(q^2-1)q^{d+n'+m'+2}$

Proof. This Lemma follows by Remark 3.5.8 from Lemma 3.5.12 with interchanged roles of n with n' and $n-m$ with m'. □

3.6. The vertices of $\tilde{\Gamma}\backslash X$

We adopt the strategy from [KMS15] to compute the quotient graph.

The group Π acts transitively on X and Ξ is the stabilizer of $x_{0,0}$ in Π. So we get $\mathcal{V}(X) = \Pi/\Xi$. In the following we consider the natural left action of Π on Y as action from the right via inversion. For the action of Π on the underlying graph Y of the Bruhat-Tits building we have two possibilities: If three does not divide d the action of Π on Y is transitive and $\tilde{\Gamma} = \Gamma$. In this case we get $\mathcal{V}(Y) = \tilde{\Gamma}\backslash\Pi$. The second case is that three is a divisor of d. Then Π acts type preservingly on Y and we have $\tilde{\Gamma}\backslash\Pi = \tilde{Y}$, where $\tilde{Y}$ is the set of vertices in Y of the same type as the vertex $y_{0,0}$. In total we get

$$\tilde{\Gamma}\backslash X \cong \tilde{\Gamma}\backslash(\Pi/\Xi) = (\tilde{\Gamma}\backslash\Pi)/\Xi \cong \begin{cases} \tilde{Y}/\Xi & \text{if } 3|d, \\ Y/\Xi & \text{if } 3\nmid d \end{cases}.$$

By Theorem 2.6.12 the set of vertices $\{y_{n,m} \mid n \geq m \geq 0\}$ is a fundamental domain for the action of Ξ on Y. Therefore the sets $\{y_{n,m} \mid n \geq m \geq 0, n+m \equiv 0 \pmod 3\}$ and $\{y_{n,m} \mid n \geq m \geq 0\}$ form a system of representatives for the Ξ-orbits on $\tilde{Y}$, respective Y. Now we label a $\tilde{\Gamma}$-orbit on X with $X_{n,m}$ if and only if it corresponds to the Ξ-orbit on Y containing the vertex $y_{n,m}$.

3.7. The edges of $\tilde{\Gamma}\backslash X$

Next we want to describe the number of edges in the quotient graph between the orbits $X_{n,m}$ and $X_{n',m'}$.

Proposition 3.7.1. *The number of edges between the orbits* $X_{n,m}$ *and* $X_{n',m'}$ *in the quotient graph* $\tilde{\Gamma}\backslash X$ *equals* $|H_{n',m'}\backslash\Upsilon_{n',m',n,m}/H_{n,m}|$.

Proof. Let x be a vertex in the $\tilde{\Gamma}$-orbit $X_{n,m}$. Then there exists an element $g \in \Pi$ such that x corresponds to the double coset $\tilde{\Gamma}g\Xi$ and $g(x_{0,0}) = x$ and $g^{-1}(y_{0,0}) = y_{n,m}$. This is because the correspondence between X and Π/Ξ is given by $x \mapsto g\Xi$ for some $g \in \Pi$

with $g(x_{0,0}) = x$ and, moreover, the correspondence of $\tilde{\Gamma}\backslash\Pi$ to either $\tilde{Y}$ or Y is given by $y_{n,m} \mapsto \tilde{\Gamma}g^{-1}$, where $g \in \Pi$ with $(y_{0,0})g = g^{-1}(y_{0,0}) = y_{n,m}$. Similarly, for each vertex $x' \in X_{n',m'}$ there exists an element $g' \in \Pi$ with $g'(x_{0,0}) = x'$ and $g'(y_{n',m'}) = y_{0,0}$.

Assume the vertex $x \in X_{n,m}$ is adjacent to the vertex $x' \in X_{n',m'}$. Then the vertex $z := (g')^{-1}(x)$ is adjacent to $x_{0,0}$ and we have $(g')^{-1}g((x_{0,0}, y_{n,m})) = (z, y_{n',m'})$. This yields, by definition, that the element $h := (g')^{-1}g$ is an element of the set $\Upsilon_{n',m',n,m}$.

Two elements $h_1 := (g')^{-1}g_1$ and $h_2 := (g')^{-1}g_2$ in $\Upsilon_{n',m',n,m}$ determine the same neighbor of the vertex x' if and only if they are in the same left coset $h_1H_{n,m} = h_2H_{n,m}$ in $\Upsilon_{n',m',n,m}/H_{n,m}$.

This is because h_1 and h_2 determine the same neighbor of x' if and only if

$$h_i((x_{0,0}, y_{n,m})) = (g')^{-1}g_i((x_{0,0}, y_{n,m})) = (z, y_{n',m'})$$

for $i = 1$ and $i = 2$. This is equivalent to

$$((g')^{-1}g_1)^{-1}(g')^{-1}g_2((x_{0,0}, y_{n,m})) = (x_{0,0}, y_{n,m}),$$

which means

$$h_1^{-1}h_2 = (g^{-1}g_1')^{-1}g^{-1}g_2' \in H_{n,m},$$

because $H_{n,m}$ is the stabilizer of the pair $(x_{0,0}, y_{n,m})$. Now $h_1^{-1}h_2 \in H_{n,m}$ holds if and only if h_1 and h_2 lie in the same left coset of $H_{n,m}$.

Next we want to consider the orbits of the stabilizer $\tilde{\Gamma}_{x'} = \Pi_{x',y_{0,0}}$ on the neighbors of the vertex x' in the orbit $X_{n,m}$. Instead of this, we will consider the orbits of the stabilizer

$$(g')^{-1}\tilde{\Gamma}_{x'}g' = (g')^{-1}\Pi_{x',y_{0,0}}g' = \Pi_{(g')^{-1}(x'),(g')^{-1}(y_{0,0})} = \Pi_{x_{0,0},y_{n',m'}} = H_{n',m'}$$

on the neighbors of $x_{0,0} = (g')^{-1}(x')$ in the orbit $(g')^{-1}X_{n,m}$.

Claim: Two neighbors z and z' of $x_{0,0}$ in $(g')^{-1}H_{n,m}$ are in the same $H_{n',m'}$-orbit if and only if the double cosets with representatives h_z and $h_{z'}$ in $H_{n',m'}\backslash\Upsilon_{n',m',n,m}/H_{n,m}$ are equal, where $h_z = (g')^{-1}g$ is the element corresponding to z and $h_{z'}$ is the element corresponding to z'.

If z and z' are in the same $H_{n',m'}$-orbit, there exists an element $\alpha \in H_{n',m'}$ with $\alpha(z) = z'$. Hence $h_{z'}^{-1}\alpha h_z((x_{0,0}, y_{n,m})) = (x_{0,0}, y_{n,m})$. Since $H_{n,m}$ is the stabilizer of the pair $(x_{0,0}, y_{n,m})$ this yields $h_{z'}^{-1}\alpha h_z \in H_{n,m}$, i.e. $h_{z'}H_{n,m} = \alpha h_z H_{n,m}$. With $\alpha \in H_{n',m'}$ it follows

$$H_{n',m'}h_{z'}H_{n,m} = H_{n',m'}\alpha h_z H_{n,m} = H_{n',m'}h_z H_{n,m}.$$

Conversely, if we have $H_{n',m'}h_{z'}H_{n,m} = H_{n',m'}h_zH_{n,m}$, then there exist $a_z, a_{z'} \in H_{n',m'}$ and $b_z, b_{z'} \in H_{n,m}$ with $a_{z'}h_{z'}b_{z'} = a_zh_zb_z$. Therefore $h_{z'} = a_{z'}^{-1}a_zh_zb_zb_{z'}^{-1}$. It follows

$$z' = h_{z'}(x_{0,0}) = a_{z'}^{-1}a_zh_zb_zb_{z'}^{-1}(x_{0,0}) \overset{H_{n,m}\subseteq\Pi_{x_{0,0}}}{=} a_{z'}^{-1}a_zh_z(x_{0,0}) = a_{z'}^{-1}a_z(z),$$

whence z and z' are in the same $H_{n',m'}$-orbit. □

Remark 3.7.2. The size of a double coset with representative $g \in \Upsilon_{n',m',n,m}$ is given by $|H_{n',m'}gH_{n,m}| = \frac{|H_{n',m'}||H_{n,m}|}{|g^{-1}H_{n',m'}g\cap H_{n,m}|} = \frac{|H_{n',m'}||H_{n,m}|}{|gH_{n,m}g^{-1}\cap H_{n',m'}|}$. According to Remark 2.6.8 we know the cardinalities of the two stabilizers $H_{n',m'}$ and $H_{n,m}$. Thus we need to compute the cardinality of the intersection $g^{-1}H_{n',m'}g \cap H_{n,m}$ in order to find the size of the corresponding double coset. By multiplication with g from the left we have $|g^{-1}H_{n',m'}g \cap H_{n,m}| = |H_{n',m'}g \cap gH_{n,m}|$. Sometimes we use this fact to compute the cardinality of the intersection $g^{-1}H_{n',m'}g \cap H_{n,m}$.

Remark 3.7.3. We consider again the last case in [KMS15] to obtain how many double cosets of which length exist in this case: Now $k = \mathbb{F}_q$ denotes the finite field with q elements and we work with the rational function field $k(t)$. We have lattices in $k(t)^2$ and we denote by x_0 the lattice class corresponding to the standard lattice in the Bruhat-Tits tree X (corresponding to the valuation ν_p) and in the Bruhat-Tits tree Y associated to the valuation ν_∞ we write y_i, for $i \in \mathbb{N}_0$, for the lattice class corresponding to $\langle t^ie_1, e_2\rangle_{\mathcal{O}_\infty}$, where $\{e_1, e_2\}$ denotes the standard basis of $k(t)^2$. Then we have the set of elements in $\Pi = \mathrm{PGL}_2(\mathcal{O}_{\{p,\infty\}})$ which map x_0 to a neighbor and y_m to y_n given by

$$\Upsilon_{n,m} := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mid \begin{array}{c} \alpha,\beta,\gamma,\delta \in k[t]; deg(\alpha) \leq \frac{d+n-m}{2}, deg(\beta) \leq \frac{d+n+m}{2}, \\ deg(\gamma) \leq \frac{d-n-m}{2}, deg(\delta) \leq \frac{d-n+m}{2}; \alpha\delta - \gamma\beta = \lambda f, \lambda \in k^\times \end{array} \right\}.$$

With this notation from [KMS15] we consider again the case $n = 0 = m$: Remember that the degree d of the place p necessarily has to be even since otherwise $\Upsilon_{0,0} = \emptyset$. Then the stabilizer of y_0 in $\mathrm{PGL}_2(\mathcal{O}_{\{\infty\}})$ is equal to $\mathrm{PGL}_2(k)$. Suppose $d = 2$. Due to the result in [KMS15] the cardinality of the set $\Upsilon_{0,0}$ is given by

$$|\Upsilon_{0,0}| = q^d(q+1)(q-1)^2$$

and we know that there exists only one double coset in this case. Thus we compute the size of the intersection $g^{-1}\,\mathrm{PGL}_2(k)g \cap \mathrm{PGL}_2(k)$ as follows

$$|g^{-1}\,\mathrm{PGL}_2(k)g \cap \mathrm{PGL}_2(k)| = \frac{|\,\mathrm{PGL}_2(k)|^2}{|\Upsilon_{0,0}|} = \frac{q^2(q+1)^2(q-1)^2}{q^2(q+1)(q-1)^2} = q+1.$$

Now suppose the degree d of the place p is greater or equal to 4. Then by the result in [KMS15] we have

$$|\Upsilon_{0,0}| = q^d(q+1)(q-1)^2.$$

Furthermore, the number of double cosets is

$$|\mathrm{PGL}_2(k)\backslash\Upsilon_{0,0}/\mathrm{PGL}_2(k)| = \tfrac{q(q^{d-3}+1)}{q+1}.$$

Take $M := \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \Upsilon_{0,0}$. We choose a representative $g := \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ of the double coset $\mathrm{PGL}_2(k)M\,\mathrm{PGL}_2(k)$ with $deg(\alpha) = deg(\delta) = \frac{d}{2}$, $deg(\gamma) \leq deg(\beta) < \frac{d}{2}$, α, δ are monic polynomials and $det(g) = f$. Now we calculate

$$g^{-1}\,\mathrm{PGL}_2(k)g = \left\{\begin{pmatrix}\frac{\delta}{f} & -\frac{\beta}{f}\\ -\frac{\gamma}{f} & \frac{\alpha}{f}\end{pmatrix}\begin{pmatrix}a_1 & a_2\\ a_3 & a_4\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \mid \begin{pmatrix}a_1 & a_2\\ a_3 & a_4\end{pmatrix} \in \mathrm{PGL}_2(k)\right\}$$
$$= \left\{\begin{pmatrix}\frac{\delta(a_1\alpha+a_2\gamma)-\beta(a_3\alpha+a_4\gamma)}{f} & \frac{\delta(a_1\beta+a_2\delta)-\beta(a_3\beta+a_4\delta)}{f}\\ \frac{-\gamma(a_1\alpha+a_2\gamma)+\alpha(a_3\alpha+a_4\gamma)}{f} & \frac{-\gamma(a_1\beta+a_2\delta)+\alpha(a_3\beta+a_4\delta)}{f}\end{pmatrix} \mid \begin{pmatrix}a_1 & a_2\\ a_3 & a_4\end{pmatrix} \in \mathrm{PGL}_2(k)\right\}.$$

Next we intersect $g^{-1}\,\mathrm{PGL}_2(k)g$ with $\mathrm{PGL}_2(k)$. Then the entries of a matrix in this intersection have to be in the field k, which implies f divides

$\delta(a_1\alpha + a_2\gamma) - \beta(a_3\alpha + a_4\gamma) = a_1 f + (a_1 - a_4)\gamma\beta - a_3\alpha\beta + a_2\gamma\delta$, where we use the determinant of g for this equation. Now we deduce that f has to divide

$(a_1-a_4)\gamma\beta-a_3\alpha\beta+a_2\gamma\delta$. For degree reasons we obtain $(a_1-a_4)\gamma\beta-a_3\alpha\beta+a_2\gamma\delta = 0$. If $deg(\beta) \neq deg(\gamma)$ it follows $a_2 = 0 = a_3$ and $a_1 = a_4$. So we have $q-1$ choices for $a_1 = a_4 \in k^{\times}$. In the case $deg(\beta) = deg(\gamma) = m$ we derive $a_3\beta_m = a_2\gamma_m$, where β_m is the leading coefficient of β and γ_m is the leading coefficient of γ. Hence, if we choose for example the first column with entries a_1 and a_3, the other two entries are given by the equations. This implies $q^2 - 1$ choices for this first column. In total we get $|g^{-1}\,\mathrm{PGL}_2(k)g \cap \mathrm{PGL}_2(k)| = \begin{cases}\frac{q-1}{q-1} = 1 \text{ if } deg(\beta) \neq deg(\gamma)\\ \frac{q^2-1}{q-1} = q+1 \text{ if } deg(\beta) = deg(\gamma)\end{cases}$, which implies $|\mathrm{PGL}_2(k)g\,\mathrm{PGL}_2(k)| = \begin{cases}q^2(q+1)^2(q-1)^2 \text{ if } deg(\beta) \neq deg(\gamma)\\ q^2(q+1)(q-1)^2 \text{ if } deg(\beta) = deg(\gamma)\end{cases}$ as possible length for the double coset. Let x denote the number of double cosets with length $q^2(q+1)^2(q-1)^2$ and y denote the number of double cosets with length $q^2(q+1)(q-1)^2$. From [KMS15] we know

$$|\Upsilon_{0,0}| = (q-1)^2(q+1)q^d = q^2(q+1)(q-1)^2(x(q+1)+y) \text{ and hence}$$
$$x(q+1) + y = q^{d-2}.$$

Furthermore, we deduce from [KMS15] that $x + y = \frac{q(q^{d-3}+1)}{q+1}$. These two equations for x and y yield

$$x = q^{d-3} - \tfrac{q^{d-3}+1}{q+1} = \tfrac{q^{d-2}-1}{q+1} \text{ and } y = q^{d-2} - (q+1)q^{d-3} + q^{d-3} + 1 = 1.$$

So there is only one double coset of length $q^2(q+1)(q-1)^2$ and all the other double cosets have length $q^2(q+1)^2(q-1)^2$.

Remark 3.7.4. Consider the map τ given by $\begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix}^\tau = \begin{pmatrix} \iota & \theta & \gamma \\ \rho & \varepsilon & \beta \\ \vartheta & \delta & \alpha \end{pmatrix}$. From Remark 3.5.8 we know that $|\Upsilon_{n',m',n,m}| = |\Upsilon^\tau_{n,n-m,n',n'-m'}|$. Furthermore, we find for the stabilizers of the vertices $y_{n,m}$ that $|H_{n,m}| = |H^\tau_{n,n-m}|$.

Now for 3×3 matrices A, M, B we have $(AMB)^\tau = B^\tau M^\tau A^\tau$. This holds because we can write $M^\tau = UM^TU$, where T is the transposition of matrices and $U = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Then it follows $(AMB)^\tau = U(AMB)^TU = UB^TM^TA^TU = UB^TUUM^TUUA^TU = B^\tau M^\tau A^\tau$. We conclude that for a double coset with representative g the following holds:

$$|(H_{n',m'}gH_{n,m})^\tau| = |H^\tau_{n,n-m}g^\tau H^\tau_{n',n'-m'}|.$$

Moreover, we have

$$|(g^{-1}H_{n',m'}g \cap H_{n,m})^\tau| = |g^\tau H^\tau_{n',n'-m'}g^{-\tau} \cap H^\tau_{n,n-m}|.$$

Now let $\mathcal{C}$ be a system of representatives, then the equality

$$\textstyle\sum_{g \in \mathcal{C}} \frac{|H_{n',m'}||H_{n,m}|}{|g^{-1}H_{n',m'}g \cap H_{n,m}|} = |\Upsilon_{n',m',n,m}| = |\Upsilon^\tau_{n,n-m,n',n'-m'}| = \\ \textstyle\sum_{g^\tau \in \mathcal{C}^\tau} \frac{|H^\tau_{n',n'-m'}||H^\tau_{n,n-m}|}{|g^\tau H^\tau_{n',n'-m'}g^{-\tau} \cap H^\tau_{n,n-m}|}$$

holds. Therefore we have the following symmetry: If we know the number of double cosets for a case n', m', n, m, then we can easily compute the number of double cosets in the case n', $n'-m'$, n, $n-m$. To do this we can use the solution for the case n', m', n, m and interchange the roles of n' and n as well as the roles of $n-m$ and m' to obtain the solution for the case n', $n'-m'$, n, $n-m$.

3.7.1. The case $\kappa < 0$, i.e. $d < 2n + n' - m + m'$

In the next three subsections we calculate the number of edges between two given orbits. By Proposition 3.7.1 we can compute the number of the corresponding double cosets to obtain the number of edges between two orbits. In order to find this number we distinguish between three cases, in particular we consider the cases $\kappa < 0$, $\kappa = 0$ and $\kappa > 0$. In all three cases we have to work with several subcases corresponding to the different stabilizers (see 2.6.9) in the cases $n = m = 0$, $n > m = 0$, $n = m > 0$, $n > m > 0$ and similarly for the indices n' and m'.

From now on let $\Upsilon_{n',m',n,m} \neq \emptyset$ and $M = \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix} \in \Upsilon_{n',m',n,m}$.

Suppose $\kappa < 0$, which is equivalent to $d < 2n + n' - m + m'$. We deduce from the degree restraints in the definition of $\Upsilon_{n',m',n,m}$ (cf. 3.5.2) that

$$\vartheta = 0,\ deg(\alpha) < n',\ deg(\beta) < n + n' - m,\ deg(\gamma) < n + n',\ deg(\delta) < m',\ deg(\varepsilon) < n - m + m',\ deg(\theta) < n + m',\ deg(\rho) < n - m,\ deg(\iota) < n.$$

Now for $n = m = 0$ it follows $\vartheta = 0 = \rho = \iota$, which is a contradiction to $det(M) = \lambda f \neq 0$. Similarly, for $n' = m' = 0$ the entries in the first column of M are equal to zero, i.e. $\alpha = 0 = \delta = \vartheta$, what is again a contradiction to $det(M) = \lambda f \neq 0$. In the following we distinguish between three possible cases for n and m, in particular $n = m > 0$, $n > m = 0$ and $n > m > 0$, and in each case we have the analogous cases for n' and m', i.e. $n' = m' > 0$, $n' > m' = 0$ and $n' > m' > 0$.

1. $n = m > 0$:

   a) $n' > m' = 0$: In this case we would have $\vartheta = 0 = \rho = \delta = \varepsilon$, which yields $det(M) = 0 \neq \lambda f$ for any $\lambda \in k^\times$. So this case does not occur.

   b) $n' = m' > 0$: Because of $deg(\rho) < n - m = 0$ and $deg(\vartheta) \leq \kappa < 0$ it is $det(M) = \iota(\alpha\varepsilon - \delta\beta) = \lambda f$. Since f is irreducible there are two possibilities, i.e.

   $$(\iota \in k^\times,\ \alpha\varepsilon - \delta\beta = \mu f \text{ for some } \mu \in k^\times) \text{ or } (\iota = \mu f, \text{ for } \mu \in k^\times, \text{ and } \alpha\varepsilon - \delta\beta \in k^\times).$$

   We start with $\iota \in k^\times$, $\alpha\varepsilon - \delta\beta = \mu f$ for some $\mu \in k^\times$. Then $d = deg(\mu f) = deg(\alpha\varepsilon - \delta\beta) \leq \frac{2d-2n+n'+m'}{3}$ implies $d \leq n' + m' - 2n = 2n' - 2n$. Moreover, $\iota \in k^\times$ yields $0 = deg(\iota) \leq \frac{d+n-n'+m-m'}{3} = \frac{d+2n-2n'}{3}$ and hence $d \geq 2n' - 2n$. Therefore $d = 2n' - 2n$. For the degree restraints of the entries of M we have $deg(\alpha) \leq n' - n = \frac{d}{2}$, $deg(\beta) \leq n' - n = \frac{d}{2}$, $deg(\gamma) \leq n'$, $deg(\delta) \leq n' - n = \frac{d}{2}$, $deg(\varepsilon) \leq n' - n = \frac{d}{2}$, $deg(\theta) \leq n'$, $deg(\iota) = 0$. As a representative for $H_{n',m'}MH_{n,m}$ we may choose the following matrix: $g = \begin{pmatrix} \alpha & \beta & 0 \\ \delta & \varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with $det(g) = f$ (multiply with a suitable matrix from $H_{n',m'}$). Next we want to calculate the size of the double coset. Therefore we need to know the intersection $gH_{n,n}g^{-1} \cap H_{n',n'}$. We compute

   $$gH_{n,n}g^{-1} =$$
   $$\left\{ \begin{pmatrix} \alpha & \beta & 0 \\ \delta & \varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & a_7 \end{pmatrix} \begin{pmatrix} \frac{\varepsilon}{f} & -\frac{\beta}{f} & 0 \\ -\frac{\delta}{f} & \frac{\alpha}{f} & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & a_7 \end{pmatrix} \in H_{n,n} \right\} =$$
   $$\left\{ \begin{pmatrix} a_1\alpha + a_4\beta & a_2\alpha + a_5\beta & a_3\alpha + a_6\beta \\ a_1\delta + a_4\varepsilon & a_2\delta + a_5\varepsilon & a_3\delta + a_6\varepsilon \\ 0 & 0 & a_7 \end{pmatrix} \begin{pmatrix} \frac{\varepsilon}{f} & -\frac{\beta}{f} & 0 \\ -\frac{\delta}{f} & \frac{\alpha}{f} & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & a_7 \end{pmatrix} \in H_{n,n} \right\} =$$

$$\left\{\begin{pmatrix} \frac{\varepsilon(a_1\alpha+a_4\beta)-\delta(a_2\alpha+a_5\beta)}{f} & \frac{-\beta(a_1\alpha+a_4\beta)+\alpha(a_2\alpha+a_5\beta)}{f} & a_3\alpha+a_6\beta \\ \frac{\varepsilon(a_1\delta+a_4\varepsilon)-\delta(a_2\delta+a_5\varepsilon)}{f} & \frac{-\beta(a_1\delta+a_4\varepsilon)+\alpha(a_2\delta+a_5\varepsilon)}{f} & a_3\delta+a_6\varepsilon \\ 0 & 0 & a_7 \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & a_7 \end{pmatrix} \in H_{n,n}\right\}.$$

Intersect $gH_{n,n}g^{-1}$ with $H_{n',n'}$: Let $h := \begin{pmatrix} \alpha & \beta \\ \delta & \varepsilon \end{pmatrix}$. We know from the definition of $H_{n,n}$ that $\begin{pmatrix} a_1 & a_2 \\ a_4 & a_5 \end{pmatrix}$ is an element of $\mathrm{PGL}_2(k)$. From our computations of $gH_{n,n}g^{-1}$ we see that the upper left 2×2 block is nothing else than $h\,\mathrm{PGL}_2(k)h^{-1}$. If we intersect $gH_{n,n}g^{-1}$ with $H_{n',n'}$ we have to intersect $h\,\mathrm{PGL}_2(k)h^{-1}$ with $\mathrm{PGL}_2(k)$ for the upper left 2×2 block. According to Remark 3.7.3 we have $|h\,\mathrm{PGL}_2(k)h^{-1}\cap \mathrm{PGL}_2(k)| = q+1$ if $d=2$ and for $d\geq 4$ there are two possibilities, in particular $|h\,\mathrm{PGL}_2(k)h^{-1}\cap \mathrm{PGL}_2(k)| = \begin{cases} 1 & \text{for } \frac{q^{d-2}-1}{q+1} \text{ double cosets} \\ q+1 & \text{for 1 double coset} \end{cases}$. For the last column in $gH_{n,n}g^{-1}\cap H_{n',n'}$ there is $a_7\in k^\times, a_3, a_6\in k[t]$ with $deg(a_3)\leq n,\ deg(a_6)\leq n$. So we have $(q-1)q^{2n+2}$ possibilities for the last column. In total we have

- for $d=2$: $|gH_{n,n}g^{-1}\cap H_{n',n'}| = (q+1)\cdot(q-1)q^{2n+2} = (q^2-1)q^{2n+2}$
- for $d\geq 4$: $|gH_{n,n}g^{-1}\cap H_{n',n'}| = \begin{cases} (q-1)q^{2n+2} & \text{for } \frac{q^{d-2}-1}{q+1} \text{ double cosets} \\ (q^2-1)q^{2n+2} & \text{for 1 double coset} \end{cases}$.

Since $\kappa+n-m=\kappa<0$ we can apply Lemma 3.5.9 to obtain $|\Upsilon_{n',m',n,m}| = (q-1)^2(q^d+q^{d-1})(q^2-q)q^{2n'+2}$. By 2.6.8 the cardinality of $H_{n,n}$ is given by $(q^2-1)(q^2-q)q^{2n+2}$ and similarly for $H_{n',n'}$. It follows $\frac{|\Upsilon_{n',m',n,m}|}{|H_{n,n}||H_{n',n'}|} = \frac{(q-1)^2(q^d+q^{d-1})(q^2-q)q^{2n'+2}}{(q^2-1)(q^2-q)q^{2n+2}(q^2-1)(q^2-q)q^{2n'+2}} = \frac{q^{d-2n-4}}{(q^2-1)}$.

Therefore,

- for $d=2$:
  $\frac{|\Upsilon_{n',m',n,m}|}{|H_{n,n}||H_{n',n'}|}\cdot|gH_{n,n}g^{-1}\cap H_{n',n'}| = \frac{q^{-2n-2}}{(q^2-1)}\cdot(q^2-1)q^{2n+2} = 1$. In particular, there is only one double coset (cf. 1(x)i).
- for $d\geq 4$:
  $\frac{|\Upsilon_{n',m',n,m}|}{|H_{n,n}||H_{n',n'}|} = \frac{q^{d-2n-4}}{(q^2-1)} = \frac{q^{d-2}-1}{q+1}\cdot\frac{1}{(q-1)q^{2n+2}} + 1\cdot\frac{1}{(q^2-1)q^{2n+2}}$. Whence in this case we have $\frac{q^{d-2}-1}{q+1}+1 = \frac{q(q^{d-3}+1)}{q+1}$ double cosets (cf. 1(y)i and 2(x)i).

The second possibility is $\iota=\mu f$ for $\mu\in k^\times$ and $\alpha\varepsilon-\delta\beta\in k^\times$. Then we have $d = deg(\mu f) = deg(\iota) \leq \frac{d+n-n'+m-m'}{3} = \frac{d+2n-2n'}{3}$, which implies $d\leq n-n'$. Together with $\alpha\varepsilon-\delta\beta\in k^\times$ and $deg(\alpha\varepsilon-\delta\beta)\leq\frac{2d+2n'-2n}{3}$ it follows $d = n - n'$. For the degree restraints this means $deg(\alpha) \leq 0$, $deg(\beta) \leq 0$, $deg(\gamma) \leq n$, $deg(\delta) \leq 0$, $deg(\varepsilon) \leq 0$, $deg(\theta) \leq n$, $deg(\iota) \leq d$.

Therefore

$$\begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & f \end{pmatrix} \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ 0 & 0 & 1 \end{pmatrix} \in H_{n',n'} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & f \end{pmatrix} H_{n,n}.$$

From this equation we conclude that there is only one double coset in this case (cf. 1(a)ii and 2(a)ii).

c) $n' > m' > 0$: Now we deduce $\vartheta = 0 = \rho$ from the degree restraints for these two entries of M. Moreover, $det(M) = \iota(\alpha\varepsilon - \delta\beta) = \lambda f$. By the irreducibility of f, there are the cases

$(\iota = \mu f$, for $\mu \in k^\times$ and $\alpha\varepsilon - \delta\beta \in k^\times)$ or $(\iota \in k^\times$ and $\alpha\varepsilon - \delta\beta = \mu f$ for some $\mu \in k^\times)$.

Assume $\iota = \mu f$, for $\mu \in k^\times$, and $\alpha\varepsilon - \delta\beta \in k^\times$. Then $d = deg(\iota) \leq \frac{d+n-n'+m-m'}{3}$ implies $2d \leq n - n' + m - m'$. Using the degree restraints for the entries of M, we deduce $deg(\varepsilon) \leq \frac{-n'+m}{2} < 0$ and $deg(\delta) \leq \frac{-n'+m}{2} < 0$, i.e. $\varepsilon = 0 = \delta$, but this means $0 = \alpha\varepsilon - \delta\beta \in k^\times$, which is a contradiction.

So let $\iota \in k^\times$ and $\alpha\varepsilon - \delta\beta = \mu f$ for some $\mu \in k^\times$. This yields $d = deg(\alpha\varepsilon - \delta\beta) \leq \frac{2d+n'-n-m+m'}{3}$ and we get $d \leq n' - n - m + m'$. Moreover, we know that ι is an element in $k^\times$, hence from the degree restraint for ι we derive $d \geq n' - n - m + m'$. Therefore, we have $d = n' - n - m + m'$. Now the following degree restraints follow: $deg(\alpha) \leq n' - n$, $deg(\beta) \leq n' - n$, $deg(\gamma) \leq n'$, $deg(\delta) \leq m' - n$, $deg(\varepsilon) \leq m' - n$, $deg(\theta) \leq m'$, $deg(\iota) = 0$. Because of these inequalities we conclude $m' \geq n$, because otherwise we have $\varepsilon = 0 = \delta$, which means $\alpha\varepsilon - \delta\beta = 0$ and this is a contradiction to $det(M) \neq 0$.

First consider the case $m' = n$. Here we have $d = n' - n$ and $deg(\alpha) \leq d$, $deg(\beta) \leq d$, $deg(\gamma) \leq n'$, $deg(\delta) \leq 0$, $deg(\varepsilon) \leq 0$, $deg(\theta) \leq m'$. We may multiply with a suitable matrix in $H_{n,n}$ in order to get $\delta = 0$, $\alpha = f$ and $\varepsilon = \mu \in k^\times$. We calculate

$$\begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \varepsilon & \theta \\ 0 & 0 & \iota \end{pmatrix} = \begin{pmatrix} 1 & \beta & \gamma \\ 0 & \varepsilon & \theta \\ 0 & 0 & \iota \end{pmatrix} \begin{pmatrix} f & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in H_{n',m'} \begin{pmatrix} f & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} H_{n,n}.$$

We conclude that there is only one double coset in this case (cf. 1(a)i and 2(a)i).

Let $m' > n$, i.e. $d > n' - n$. In this case we may choose the following matrix as a representative for $H_{n',m'} M H_{n,m}$: $g = \begin{pmatrix} \alpha & \beta & 0 \\ \delta & \varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (multiply with a suitable matrix from $H_{n',m'}$). Moreover, we may assume $det(g) = \alpha\varepsilon - \delta\beta = f$. As in the case $n' = m' > 0$ we get

$$gH_{n,n}g^{-1} =$$

$$\left\{ \begin{pmatrix} \alpha & \beta & 0 \\ \delta & \varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & a_7 \end{pmatrix} \begin{pmatrix} \frac{\varepsilon}{f} & -\frac{\beta}{f} & 0 \\ -\frac{\delta}{f} & \frac{\alpha}{f} & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & a_7 \end{pmatrix} \in H_{n,n} \right\} =$$

$$\left\{ \begin{pmatrix} \frac{\varepsilon(a_1\alpha + a_4\beta) - \delta(a_2\alpha + a_5\beta)}{f} & \frac{-\beta(a_1\alpha + a_4\beta) + \alpha(a_2\alpha + a_5\beta)}{f} & a_3\alpha + a_6\beta \\ \frac{\varepsilon(a_1\delta + a_4\varepsilon) - \delta(a_2\delta + a_5\varepsilon)}{f} & \frac{-\beta(a_1\delta + a_4\varepsilon) + \alpha(a_2\delta + a_5\varepsilon)}{f} & a_3\delta + a_6\varepsilon \\ 0 & 0 & a_7 \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & a_7 \end{pmatrix} \in H_{n,n} \right\}.$$

In $gH_{n,n}g^{-1} \cap H_{n',m'}$ the following holds: $\frac{\varepsilon(a_1\delta + a_4\varepsilon) - \delta(a_2\delta + a_5\varepsilon)}{f} = 0$ and f divides $a_1\alpha\varepsilon + a_4\beta\varepsilon - a_2\alpha\delta - a_5\beta\delta = (a_1 - a_5)\alpha\varepsilon + a_4\beta\varepsilon - a_2\alpha\delta + a_5 f$, where we used the determinant of g for this equation. From the first equation we deduce $\varepsilon(a_1\delta + a_4\varepsilon - a_5\delta) = a_2\delta^2$. Since f is irreducible, we know that ε and δ are coprime polynomials. Whence, the last equation yields $a_2 = 0 = a_1\delta + a_4\varepsilon - a_5\delta$ and using the same argument again, we finally get $a_4 = 0 = a_1 - a_5$ (otherwise $\varepsilon = a_4^{-1}(a_1 - a_5)\delta)$. This means the intersection can be written as the following set of matrices:

$$gH_{n,n}g^{-1} \cap H_{n',m'} =$$

$$\left\{ \begin{pmatrix} a_5 & 0 & a_3\alpha + a_6\beta \\ 0 & a_5 & a_3\delta + a_6\varepsilon \\ 0 & 0 & a_7 \end{pmatrix} \mid a_5, a_7 \in k^\times, a_3, a_6 \in k[t] \text{ with } deg(a_3) \leq n, deg(a_6) \leq n \right\}.$$

Therefore, we calculate $|gH_{n,n}g^{-1} \cap H_{n',m'}| = (q-1)q^{2n+2}$. Notice that we divide by $(q-1)$ because we are working in the projective group. Hence we see that all double cosets have the same length.

From Lemma 3.5.9 we know $|\Upsilon_{n',m',n,m}| = (q-1)^2(q^d + q^{d-1})(q^2 - q)q^{n'+m'+2}$ and by 2.6.8 we have $|H_{n,n}| = (q^2-1)(q^2-q)q^{2n+2}$ and $|H_{n',m'}| = (q-1)^2 q^{2n'+3}$. Let x denote the number of double cosets. Then

$$\frac{x}{|gH_{n,n}g^{-1} \cap H_{n',m'}|} = \frac{x}{(q-1)q^{2n+2}} = \frac{|\Upsilon_{n',m',n,m}|}{|H_{n,n}||H_{n',m'}|}$$
$$= \frac{(q-1)^2(q^d + q^{d-1})(q^2-q)q^{n'+m'+2}}{(q^2-1)(q^2-q)q^{2n+2}(q-1)^2 q^{2n'+3}} = \frac{q^{d-n'+m'-2}}{(q-1)q^{2n+2}}.$$

We deduce that we have $x = q^{d-n'+m'-2}$ double cosets if $d - n' + m' - 2 \geq 0$, i.e. if $d \geq n' - m' + 2 \geq 3$ (cf. 1(d)i and 2(d)i).

2. $n > m = 0$:

   a) $n' > m' = 0$: To compute the number of double cosets in this case we use the symmetry 3.7.4 and the solutions of case $n = m > 0$ and $n' = m' > 0$. This gives us the following solutions: We have one double coset in the case $d = n' - n$ (cf. 1(a)i and 2(a)i). Moreover, if the degree is given by $d = 2n - 2n'$, then we find that

      - if $d = 2$, there is only one double coset (cf. 1(x)ii)
      - for $d \geq 4$ there are $\frac{q(q^{d-3}+1)}{q+1}$ double cosets (cf. 1(y)ii and 2(x)ii).

   b) $n' = m' > 0$: In this case we consider three subcases, in particular, $d < 2n + n' - m - 2m'$, $d = 2n + n' - m - 2m'$ or $d > 2n + n' - m - 2m'$.

      We start to consider the case $d \leq 2n + n' - m - 2m' = 2n - n'$. This leads to the following degree restraints for the entries of M: $deg(\alpha) \leq n' - m' = 0$, $deg(\beta) \leq n + n' - m - m'$, $deg(\gamma) \leq n + n' - m' = n$, $deg(\delta) \leq 0$, $deg(\varepsilon) \leq n - m = n$, $deg(\theta) \leq n$, $deg(\rho) \leq n - m' - m = n - m'$, $deg(\iota) \leq n - m'$. Note that from the degree restraints we can exclude the first case $d < 2n + n' - m - 2m'$, because for this case it follows $\alpha = 0 = \delta = \vartheta$, what contradicts to $det(M) \neq 0$. So we continue with the case $d = 2n - n'$. From the above degree restraints we deduce $n \geq n'$, because otherwise we have with $\rho = 0 = \vartheta = \iota$ a contradiction to $det(M) \neq 0$. Via multiplication by a suitable matrix in $H_{n',n'}$ we may assume that $\delta = 0$ and $\alpha = 1$. This implies $det(M) = \lambda f = \varepsilon\iota - \rho\theta$. Next we use again a case distinction into the subcases $n = n'$ and $n > n'$.

      Suppose $n = n'$, then $d = n$ and the degree restraints imply $\rho, \iota \in k$. Multiplication with a certain matrix in $H_{n,0}$ yields $\rho = 0, \iota = \lambda$ and $\varepsilon = f$. We calculate

      $$\begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ 0 & \rho & \iota \end{pmatrix} = \begin{pmatrix} 1 & \beta & \gamma \\ 0 & f & \theta \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1 & 0 & \gamma \\ 0 & 1 & \theta \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in H_{n',n'} \begin{pmatrix} 1 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix} H_{n,0}.$$

      This means there is only one double coset in this case (cf. 1(a)i and 2(a)i).

      Let $n > n'$ and remember that $d = 2n - n'$, $\alpha = 1$ and $\delta = 0$. As a representative for the double coset of $H_{n',n'} M H_{n,0}$ we choose the matrix

$$M\begin{pmatrix}1 & -\beta & -\gamma\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & \beta & \gamma\\ 0 & \varepsilon & \theta\\ 0 & \rho & \iota\end{pmatrix}\begin{pmatrix}1 & -\beta & -\gamma\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & \varepsilon & \theta\\ 0 & \rho & \iota\end{pmatrix} =: g \text{ with}$$
$$det(g) = f.$$

With this representative we compute

$$g^{-1}H_{n',n'}g =$$
$$\left\{\begin{pmatrix}1 & 0 & 0\\ 0 & \frac{\iota}{f} & -\frac{\theta}{f}\\ 0 & -\frac{\rho}{f} & \frac{\varepsilon}{f}\end{pmatrix}\begin{pmatrix}a_1 & a_2 & a_3\\ a_4 & a_5 & a_6\\ 0 & 0 & a_7\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \varepsilon & \theta\\ 0 & \rho & \iota\end{pmatrix} \mid \begin{pmatrix}a_1 & a_2 & a_3\\ a_4 & a_5 & a_6\\ 0 & 0 & a_7\end{pmatrix} \in H_{n',n'}\right\} =$$
$$\left\{\begin{pmatrix}a_1 & a_2\varepsilon + a_3\rho & a_2\theta + a_3\iota\\ \frac{a_4\iota}{f} & \frac{a_5\varepsilon\iota+\rho(a_6\iota-a_7\theta)}{f} & \frac{a_5\theta\iota+\iota(a_6\iota-a_7\theta)}{f}\\ -\frac{a_4\rho}{f} & \frac{-a_5\varepsilon\rho+\rho(a_7\varepsilon-a_6\rho)}{f} & \frac{-a_5\theta\rho+\iota(a_7\varepsilon-a_6\rho)}{f}\end{pmatrix} \mid \begin{pmatrix}a_1 & a_2 & a_3\\ a_4 & a_5 & a_6\\ 0 & 0 & a_7\end{pmatrix} \in H_{n',n'}\right\}.$$

In $g^{-1}H_{n',n'}g \cap H_{n,0}$: It is $a_4 = 0$ and f divides $(a_7 - a_5)\varepsilon\rho - a_6\rho^2 = \rho((a_7 - a_5)\varepsilon - a_6\rho)$. For degree reasons this is only possible for $(a_7 - a_5)\varepsilon = a_6\rho$. But this leads to $a_7 = a_5$ and $a_6 = 0$, since ε and ρ are coprime polynomials, because f is irreducible. Then the intersection is given by

$$g^{-1}H_{n',n'}g \cap H_{n,0} =$$
$$\left\{\begin{pmatrix}a_1 & a_2\varepsilon + a_3\rho & a_2\theta + a_3\iota\\ 0 & a_7 & 0\\ 0 & 0 & a_7\end{pmatrix} \mid a_1, a_7 \in k^\times,\ a_2 \in k,\ a_3 \in k[t] \text{ with } deg(a_3) \leq n'\right\}.$$

We conclude $|g^{-1}H_{n',n'}g \cap H_{n,0}| = (q-1)q^{n'+2}$. Here we again divide by $(q-1)$ because we are working in the projective group.

According to Lemma 3.5.12 1.c) the cardinality of $\Upsilon_{n',n',n,0}$ is given by $(q-1)(q^2-1)^2q^{d+2n-m+2} = (q-1)(q^2-1)^2q^{d+2n+2}$. From 2.6.8 we know the cardinalities of the two stabilizers $H_{n,0}$ and $H_{n',n'}$. Therefore, we have

$$\frac{|\Upsilon_{n',n',n,0}|}{|H_{n',n'}||H_{n,0}|}|g^{-1}H_{n',n'}g \cap H_{n,0}| =$$
$$\frac{(q-1)(q^2-1)^2q^{d+2n+2}}{(q^2-1)(q^2-q)q^{2n'+2}(q^2-1)(q^2-q)q^{2n+2}}(q-1)q^{n'+2} = q^{d-n'-2}$$

double cosets if $d \geq n'+2 \geq 3$ (cf. 1(e)i and 2(e)i).

For the case $d > 2n + n' - m - 2m'$ we distinguish again three cases, in particular $d < n'-n+2m+m'$, $d = n'-n+2m+m'$ and $d > n'-n+2m+m'$. In order to solve the first two cases we can use the symmetry 3.7.4. Therefore we use the solutions above, in particular, the solutions for $n > m = 0$ and $n' = m' > 0$ with $d < 2n-n'$ or $d = 2n-n'$. We conclude that $d < n'-n+2m+m'$ is not possible. Furthermore, we have for $d = n'-n+2m+m' = 2n'-n$ that $n' > n$, because of $d > 2n+n'-m-2m' = 2n-n'$. Via symmetry 3.7.4 we find q^{d-n-2} double cosets in this case, if the degree fulfills $d \geq n+2 \geq 3$ (cf. 1(c)i and 2(c)i).

The final subcase is $d > n' - n + 2m + m' = 2n' - n$. We still have $d > 2n + n' - m - 2m' = 2n - n'$ and $d < 2n + n' - m + m' = 2n + 2n'$, hence $\kappa > -n' = -m'$ and $\kappa > -n$. Moreover, the degree restraints are given by $deg(\alpha) \leq \frac{d+n'-2n}{3}$, $deg(\beta) \leq \frac{d+n+n'}{3}$, $deg(\gamma) \leq \frac{d+n+n'}{3}$, $deg(\delta) \leq \frac{d-2n+n'}{3}$, $deg(\varepsilon) \leq \frac{d+n+n'}{3}$, $deg(\theta) \leq \frac{d+n+n'}{3}$, $deg(\rho) \leq \frac{d+n-2n'}{3}$, $deg(\iota) \leq \frac{d+n-2n'}{3}$. Due to Lemma 3.5.12 1.a) we have $|\Upsilon_{n',n',n,0}| = (q-1)^2(q^2-1)^2 q^{d+2\kappa+2n-m+n'+m'+1} = (q-1)^2(q^2-1)^2 q^{d+2\kappa+2n+2n'+1}$.

By multiplication with suitable matrices of $H_{n',n'}$ from the left and suitable matrices of $H_{n,0}$ from the right we may choose as representative for the double coset the matrix $g := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ 0 & \rho & \iota \end{pmatrix}$, such that $\alpha, \varepsilon, \iota$ have their respective maximal possible degree and all the other entries have not their respective maximal possible degree according to the degree restraints for the entries of M.

It follows

$$H_{n',n'}g = \left\{ \begin{pmatrix} a_1\alpha + a_2\delta & a_1\beta + a_2\varepsilon + a_3\rho & a_1\gamma + a_2\theta + a_3\iota \\ a_4\alpha + a_5\delta & a_4\beta + a_5\varepsilon + a_6\rho & a_4\gamma + a_5\theta + a_6\iota \\ 0 & a_7\rho & a_7\iota \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & a_7 \end{pmatrix} \in H_{n',n'} \right\}$$

and

$$gH_{n,0} = \left\{ \begin{pmatrix} b_1\alpha & b_2\alpha + b_4\beta + b_6\gamma & b_3\alpha + b_5\beta + b_7\gamma \\ b_1\delta & b_2\delta + b_4\varepsilon + b_6\theta & b_3\delta + b_6\varepsilon + b_7\theta \\ 0 & b_4\rho + b_6\iota & b_5\rho + b_7\iota \end{pmatrix} \mid \begin{pmatrix} b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 \\ 0 & b_6 & b_7 \end{pmatrix} \in H_{n,0} \right\}.$$

Therefore the intersection $H_{n',n'}g \cap gH_{n,0}$ is determined by the following equations:

$$a_4\alpha + a_5\delta = b_1\delta,\ a_1\alpha + a_2\delta = b_1\alpha,\ (a_7 - b_4)\rho = b_6\iota,$$

$$a_4\beta + a_5\varepsilon + a_6\rho = b_2\delta + b_4\varepsilon + b_6\theta,\ a_1\beta + a_2\varepsilon + a_3\rho = b_2\alpha + b_4\beta + b_6\gamma,\ (a_7 - b_7)\iota = b_5\rho,$$

$$a_4\gamma + a_5\theta + a_6\iota = b_3\delta + b_5\varepsilon + b_7\theta,\ a_1\gamma + a_2\theta + a_3\iota = b_3\alpha + b_5\beta + b_7\gamma.$$

Now $a_4\alpha + a_5\delta = b_1\delta$ implies $a_4 = 0$ and since $\delta \neq 0$ we get additionally $a_5 = b_1$. From $a_1\alpha + a_2\delta = b_1\alpha$ we deduce $a_1 = b_1$ and $a_2 = 0$. Using $(a_7 - b_4)\rho = b_6\iota$ we obtain $b_6 = 0$ and $a_7 = b_4$ because $\rho \neq 0$. The equation $(a_7 - b_7)\iota = b_5\rho$ yields $a_7 = b_7$ and $b_5 = 0$. Because of $a_4\beta + a_5\varepsilon + a_6\rho = b_2\delta + b_4\varepsilon + b_6\theta$ we see that $a_5 = b_4$. We derive the following remaining equations:

$$a_6\rho = b_2\delta,\ a_3\rho = b_2\alpha,\ a_6\iota = b_3\delta,\ a_3\iota = b_3\alpha.$$

Since $\vartheta = 0$ and the entries of a row in g have to be coprime, we know that ρ and ι are coprime polynomials. Together with the above equations this implies that there exists some $b \in k[t]$, such that $a_3 = b\alpha$. For the degree of the polynomial b we have $deg(b) \leq deg(a_3) - deg(\alpha) \leq n' - \frac{d+n'-2n}{3} = \frac{2n'+2n-d}{3} = -\kappa$. Due to the equations we get $b_2 = b\rho$, $a_6 = b\delta$ and $b_3 = b\iota$. Hence the intersection is given by

$$H_{n',n'}g \cap gH_{n,0} = \left\{ \begin{pmatrix} a_1 & 0 & b\alpha \\ 0 & a_1 & b\delta \\ 0 & 0 & a_1 \end{pmatrix} g \mid a_1 \in k^\times, deg(b) \leq -\kappa \right\}.$$

We arrive at $|H_{n',n'}g \cap gH_{n,0}| = q^{-\kappa+1}$. Note that we divide by $q-1$ since we are working in the projective group. Thus the size of a double coset in this case is given by $|H_{n',n'}gH_{n,0}| = (q^2-1)(q^2-q)q^{2n'+2}(q^2-1)(q^2-q)q^{2n+2}q^{\kappa-1}$. We conclude there are

$$\frac{|\Upsilon_{n',n',n,0}|}{|H_{n',n'}gH_{n,0}|} = \frac{(q-1)^2(q^2-1)^2q^{d+2\kappa+2n+2n'+1}}{(q^2-1)^2(q^2-q)^2q^{2n'+2n+5+\kappa}} = q^{d+\kappa-4}$$

double cosets in this case, if the degree of f fulfills $d \geq 4 - \kappa \geq 5$ (cf. 1(h)i and 2(h)i).

c) The case $n' > m' > 0$: Now we distinguish between three subcases. The three cases are $d < 2n+n'-m-2m'$, $d = 2n+n'-m-2m'$ and $d > 2n+n'-m-2m'$.

1. case: $d < 2n+n'-m-2m'$. Here we have $deg(\delta) < 0$ and hence $\delta = 0$. So the determinant of M can be written as $det(M) = \alpha(\varepsilon\iota - \rho\theta) = \lambda f$. According to the Irreducibility of f we get

$(\varepsilon\iota - \rho\theta \in k^\times,\ \alpha = \mu f$, for some $\mu \in k^\times)$ or $(\alpha \in k^\times,\ \varepsilon\iota - \rho\theta = \mu f$, for some $\mu \in k^\times)$.

Assume $\varepsilon\iota - \rho\theta \in k^\times$ and $\alpha = \mu f$. Then $d \leq deg(\alpha) \leq \frac{d+2n'-2n+m-m'}{3}$, which implies $2d \leq 2n' - 2n + m - m'$, i.e. $d \leq n' - n - \frac{m'}{2}$. But from this we get $deg(\rho) \leq -\frac{m'}{2} < 0$ and $deg(\iota) \leq -\frac{m'}{2} < 0$ and hence $\varepsilon\iota - \rho\theta = 0$. This contradicts $\varepsilon\iota - \rho\theta \in k^\times$.

Now let $\alpha \in k^\times$ and $\varepsilon\iota - \rho\theta = \mu f$. From this we derive $d \leq deg(\varepsilon\iota - \rho\theta) \leq \frac{2d+2n-2n'+m'}{3}$. In particular, we have $d \leq 2n - 2n' + m'$. If we assume $d < 2n - 2n' + m'$, then $deg(\alpha) \leq \frac{d+2n'-2n-m'}{3} < 0$ yields $\alpha = 0$, which contradicts $\alpha \neq 0$. So we have $d = 2n - 2n' + m'$ and $deg(\alpha) \leq 0$, $deg(\beta) \leq n$, $deg(\gamma) \leq n$, $deg(\delta) \leq -n' + m' < 0$, $deg(\varepsilon) \leq n - n' + m'$, $deg(\theta) \leq n - n' + m'$, $deg(\rho) \leq n - n'$, $deg(\iota) \leq n - n'$.

We conclude that $n \geq n'$, because otherwise the degree restraints imply $\vartheta = 0 = \rho = \iota$, which contradicts $det(M) \neq 0$.

First consider the case $n = n'$. It follows $d = m'$, $deg(\varepsilon) \leq d$, $deg(\theta) \leq d$ and $\rho, \iota \in k$.

Suppose $\rho = 0$. Then $det(M) = \alpha\varepsilon\iota = \lambda f$. According to the degree restraints and the Irreducibility of f there exists a $\mu \in k^\times$ with $\varepsilon = \mu f$ and α and ι are non-zero elements in the field K. It follows

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \varepsilon & \theta \\ 0 & 0 & \iota \end{pmatrix} = \begin{pmatrix} \alpha & 0 & \gamma \\ 0 & \mu & \theta \\ 0 & 0 & \iota \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha^{-1}\beta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in$$
$$H_{n',m'} \begin{pmatrix} 1 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix} H_{n,0}.$$

If ρ is a non-zero element in k we may deduce from $\varepsilon\iota - \rho\theta = \mu f$, that $\theta = \rho^{-1}(\varepsilon\iota - \mu f)$. Now we get

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \varepsilon & \theta \\ 0 & \rho & \iota \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\rho^{-1}\mu & \varepsilon \\ 0 & 0 & \rho \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & 0 & 1 \\ 0 & 1 & \rho^{-1}\iota \end{pmatrix}$$
$$\in H_{n',m'} \begin{pmatrix} 1 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix} H_{n,0}.$$

Therefore, we obtain one double coset in this case (cf. 1(a)i and 2(a)i).

Next consider the case $n > n'$. Then $deg(\varepsilon) < d$, $deg(\theta) < d$, $deg(\rho) < d$, $deg(\iota) < d$. Due to Lemma 3.5.9 the cardinality of the set $\Upsilon_{n',m',n,0}$ is the following: $|\Upsilon_{n',m',n,0}| = (q-1)^2(q^d + q^{d-1})(q^2 - q)q^{2n+2}$. For the length of a double coset we have the formula $|H_{n',m'} g H_{n,0}| = \frac{|H_{n',m'}||H_{n,0}|}{|g^{-1}H_{n',m'}g \cap H_{n,0}|}$, where $g \in \Upsilon_{n',m',n,0}$ is a representative of the double coset. By 2.6.8 we know that $|H_{n,0}| = (q^2-1)(q^2-q)q^{2n+2}$ and $|H_{n',m'}| = (q-1)^2 q^{2n'+3}$. In order to calculate the length of a given double coset we need to know the cardinality of the intersection $|g^{-1}H_{n',m'}g \cap H_{n,0}|$. Let $g \in \Upsilon_{n',m',n,0}$ be a representative of some double coset. Then we may assume that $\alpha = 1$, $\beta = 0 = \gamma$, i.e. that $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & \theta \\ 0 & \rho & \iota \end{pmatrix}$. Moreover, we may assume $det(g) = f$. We compute

$$g^{-1}H_{n',m'}g =$$
$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\iota}{f} & -\frac{\theta}{f} \\ 0 & -\frac{\rho}{f} & \frac{\varepsilon}{f} \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & \theta \\ 0 & \rho & \iota \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{pmatrix} \in H_{n',m'} \right\} =$$

$$\left\{ \begin{pmatrix} a_1 & a_2\varepsilon + a_3\rho & a_2\theta + a_3\iota \\ 0 & \frac{a_4\iota\varepsilon + a_5\iota\rho - a_6\rho\theta}{f} & \frac{a_4\iota\theta + a_5\iota^2 - a_6\iota\theta}{f} \\ 0 & \frac{-a_4\rho\varepsilon - a_5\rho^2 + a_6\rho\varepsilon}{f} & \frac{-a_4\rho\theta - a_5\iota\rho + a_6\varepsilon\iota}{f} \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{pmatrix} \in H_{n',m'} \right\}.$$

If we intersect this set with $H_{n,0}$, we have that each entry of this matrix is a polynomial. Thus the polynomial f divides $-a_4\rho\varepsilon - a_5\rho^2 + a_6\rho\varepsilon$. Since $deg(\rho) < d = deg(f)$, it follows that f divides $-a_4\varepsilon - a_5\rho + a_6\varepsilon$. According to $deg(a_5\rho) < d$ and $deg(\varepsilon) < d$ this implies $-a_4\varepsilon - a_5\rho + a_6\varepsilon = 0$ and hence $a_5\rho = (a_6 - a_4)\varepsilon$. Moreover, we know that f divides $a_4\iota\varepsilon + a_5\iota\rho - a_6\rho\theta = a_4(f + \rho\theta) + a_5\rho\iota - a_6\rho\theta$, using the determinant of g. Since f divides $a_4 f$ we see that f divides $\rho((a_4 - a_6)\theta + a_5\iota)$. Because of $deg(\rho) < d$, $\quad deg(\theta) < d$ and $deg(a_5\iota) < d$ we get $a_5\iota = (a_6 - a_4)\theta$. If we assume that $(a_6 - a_4)$ is a non-zero element in k, then we have that a_5 is a non-zero element in $k[t]$, too. But this yields $\varepsilon = a_5(a_6 - a_4)^{-1}\rho$ and $\theta = a_5(a_6 - a_4)^{-1}\iota$, whence $\begin{pmatrix} \varepsilon \\ \theta \end{pmatrix}$ and $\begin{pmatrix} \rho \\ \iota \end{pmatrix}$ are linearly dependent, which implies that the determinant of g is zero, a contradiction to $det(g) = f$. We deduce $a_5 = 0$ and $a_4 = a_6$. Then

$$g^{-1}H_{n',m'}g \cap H_{n,0} =$$
$$\left\{ \begin{pmatrix} a_1 & a_2\varepsilon + a_3\rho & a_2\theta + a_3\iota \\ 0 & a_4 & 0 \\ 0 & 0 & a_4 \end{pmatrix} \mid a_1, a_4 \in K^\times, \quad deg(a_2) \leq n' - m', \quad deg(a_3) \leq n' \right\}.$$

We conclude $|g^{-1}H_{n',m'}g \cap H_{n,0}| = (q-1)q^{2n'-m'+2}$. Therefore we see that every double coset has the same length, because the size of the intersection $g^{-1}H_{n',m'}g \cap H_{n,0}$ is the same for every $g \in \Upsilon_{n',m',n,0}$. Let $\mathcal{C}$ be a system of representatives for the double cosets. Then $\sum_{g\in\mathcal{C}} \frac{|H_{n',m'}||H_{n,0}|}{|g^{-1}H_{n',m'}g\cap H_{n,0}|} = |\Upsilon_{n',m',n,0}|$. Hence

$$|\mathcal{C}| = \textstyle\sum_{g\in\mathcal{C}} 1 = |\Upsilon_{n',m',n,0}| \cdot \frac{|g^{-1}H_{n',m'}g\cap H_{n,0}|}{|H_{n',m'}||H_{n,0}|} = q^{d-2-m'} = q^{2n-2n'-2},$$

if we have $d \geq m' + 2 \geq 3$ (cf. 1(d)ii and 2(d)ii).

2. case: $d = 2n + n' - m - 2m'$. Now the degree restraints for the entries of M are given by $deg(\alpha) \leq n' - m'$, $deg(\beta) \leq n + n' - m'$, $deg(\gamma) \leq n + n' - m'$, $deg(\delta) \leq 0$, $deg(\varepsilon) \leq n$, $deg(\theta) \leq n$, $deg(\rho) \leq n - m'$, $deg(\iota) \leq n - m'$. Here we necessarily have $n \geq m'$, since otherwise $\rho = 0 = \iota = \vartheta$ is a contradiction to $det(M) \neq 0$.

Assume $\delta = 0$. In this case $det(M) = \alpha(\varepsilon\iota - \rho\theta)$ and similar to the above case $d < 2n + n' - m - 2m'$ we obtain $d = 2n - 2n' + m'$. Thus $2n - 2n' + m' = d = 2n + n' - m - 2m'$, but this means m' is equal to n', what is not possible since we are in the case $n' > m'$. Therefore, $\delta \neq 0$.

Next we consider the two subcases $n = m'$ and $n > m'$. We start with $n = m'$. This yields $d = n'$ and ρ and ι are elements in the field k. Whence, we can find a suitable matrix in $H_{n,0}$, such that the multiplication with this matrix yields a new matrix M, where we have $\rho = 0$ and $\iota = 1$. Furthermore, the determinant of this matrix M is given by $det(M) = \alpha\varepsilon - \delta\beta = \lambda f$. Thus $\beta = \delta^{-1}(\alpha\varepsilon - \lambda f)$. Using this Identity we compute

$$\begin{pmatrix}\alpha & \beta & \gamma\\ \delta & \varepsilon & \theta\\ 0 & 0 & \iota\end{pmatrix} = \begin{pmatrix}\delta^{-1}\lambda & \delta^{-1}\alpha & \gamma\\ 0 & 1 & \theta\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & f & 0\\ 1 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}\delta & \varepsilon & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix} \in$$
$$H_{n',m'}\begin{pmatrix}0 & f & 0\\ 1 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}H_{n,0}.$$

So there is only one double coset in this case (cf. 1(a)i and 2(a)i).

Let $n > m'$. As representative for our double coset we can assume $\varepsilon = 0 = \theta$ and $\delta = 1$ (multiply with a suitable matrix from $H_{n,0}$), and moreover, we can assume $\alpha = 0$ (multiply with a suitable matrix from $H_{n',m'}$). Furthermore, we may assume the determinant equal to f, which means $\lambda = 1$. So we take $g := \begin{pmatrix}0 & \beta & \gamma\\ 1 & 0 & 0\\ 0 & \rho & \iota\end{pmatrix}$ with $det(g) = f$ as a representative for the double coset.

We calculate

$$g^{-1}H_{n',m'}g =$$
$$\left\{\begin{pmatrix}0 & 1 & 0\\ -\frac{\iota}{f} & 0 & \frac{\gamma}{f}\\ \frac{\rho}{f} & 0 & -\frac{\beta}{f}\end{pmatrix}\begin{pmatrix}a_1 & a_2 & a_3\\ 0 & a_4 & a_5\\ 0 & 0 & a_6\end{pmatrix}\begin{pmatrix}0 & \beta & \gamma\\ 1 & 0 & 0\\ 0 & \rho & \iota\end{pmatrix} \mid \begin{pmatrix}a_1 & a_2 & a_3\\ 0 & a_4 & a_5\\ 0 & 0 & a_6\end{pmatrix} \in H_{n',m'}\right\} =$$
$$\left\{\begin{pmatrix}a_4 & a_5\rho & a_5\iota\\ -\frac{a_2\iota}{f} & \frac{-\iota(a_1\beta+a_3\rho)+a_6\rho\gamma}{f} & \frac{-\iota(a_1\gamma+a_3\iota)+a_6\iota\gamma}{f}\\ \frac{a_2\rho}{f} & \frac{\rho(a_1\beta+a_3\rho)-a_6\beta\rho}{f} & \frac{\rho(a_1\gamma+a_3\iota)-a_6\beta\iota}{f}\end{pmatrix} \mid \begin{pmatrix}a_1 & a_2 & a_3\\ 0 & a_4 & a_5\\ 0 & 0 & a_6\end{pmatrix} \in H_{n',m'}\right\}.$$

In $g^{-1}H_{n',m'}g \cap H_{n,0}$: The entries of a matrix in this intersection are necessarily polynomials. Hence $a_2 = 0$ and f divides $\rho(a_1\beta + a_3\rho - a_6\beta)$. According to the Irreducibility of f and for degree reasons it follows that the condition can only be fulfilled if $(a_1 - a_6)\beta + a_3\rho = 0$, i.e. $(a_6 - a_1)\beta = a_3\rho$. Since f is irreducible, the polynomials ρ and β are coprime. Together with the fact that either ρ or β has to be a polynomial of maximal possible degree the above equation leads to $a_6 = a_1$ and $a_3 = 0$. We conclude

$$g^{-1}H_{n',m'}g \cap H_{n,0} = \left\{\begin{pmatrix}a_4 & a_5\rho & a_5\iota\\ 0 & a_1 & 0\\ 0 & 0 & a_1\end{pmatrix} \mid a_1, a_4 \in k^\times, \quad deg(a_5) \le m'\right\}.$$ Thus

$|g^{-1}H_{n',m'}g \cap H_{n,0}| = (q-1)q^{m'+1}$.

Using Lemma 3.5.12 1.b) we see that $|\Upsilon_{n',m',n,0}| = (q-1)^2(q^2-1)q^{d+\kappa+2n+n'+3}$. With 2.6.8 it follows that there are

$$\frac{|\Upsilon_{n',m',n,0}|}{|H_{n',m'}||H_{n,0}|}|g^{-1}H_{n',m'}g \cap H_{n,0}| = \frac{(q-1)^2(q^2-1)q^{d+\kappa+2n+n'+3}}{(q-1)^2q^{2n'+3}(q^2-q)(q^2-q)q^{2n+2}}(q-1)q^{m'+1} = q^{2n-2m'-2} = q^{d-n'-2},$$

double cosets, if $d \geq n'+2 \geq 4$ (cf. 1(d)ii and 2(d)ii).

3.case: $d > 2n+n'-m-2m'$. In this case we do again a case distinction.

First let $d \leq n'-n+2m+m'$. Then the degree restraints are $deg(\alpha) \leq n'-n+m = n'-n$, $deg(\beta) \leq n'$, $deg(\gamma) \leq n'+m = n'$, $deg(\delta) \leq m+m'-n = m'-n$, $deg(\varepsilon) \leq m'$, $deg(\theta) \leq m+m' = m'$, $deg(\rho) \leq 0$, $deg(\iota) \leq m = 0$. Because of $2n+n'-m-2m' < d \leq n'-n+2m+m'$ we have $n < m'$. If we multiply with a certain matrix from $H_{n,0}$ we may assume $\rho = 0$ and $\iota = 1$. For the determinant we obtain $det(M) = \alpha\varepsilon - \delta\beta = \lambda f$. This implies $d = deg(\alpha\varepsilon - \delta\beta) \leq n'-n+m'$. Together with $0 = deg(\iota) \leq \frac{d+n-n'-m'}{3}$ we have $d = n'-n+m'$.

For the double coset with representative M we choose another representative, in particular we choose $\begin{pmatrix} 1 & 0 & -\gamma \\ 0 & 1 & -\theta \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} \alpha & \beta & 0 \\ \delta & \varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} =: g$ with $det(g) = \alpha\varepsilon - \delta\beta = f$.

We calculate

$$gH_{n,0}g^{-1} = \left\{ \begin{pmatrix} \alpha & \beta & 0 \\ \delta & \varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & a_6 & a_7 \end{pmatrix}\begin{pmatrix} \frac{\varepsilon}{f} & -\frac{\beta}{f} & 0 \\ -\frac{\delta}{f} & \frac{\alpha}{f} & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & a_6 & a_7 \end{pmatrix} \in H_{n,0} \right\} =$$
$$\left\{ \begin{pmatrix} \frac{a_1\alpha\varepsilon-\delta(a_2\alpha+a_4\beta)}{f} & \frac{-a_1\alpha\beta+\alpha(a_2\alpha+a_4\beta)}{f} & a_3\alpha+a_5\beta \\ \frac{a_1\delta\varepsilon-\delta(a_2\delta+a_4\varepsilon)}{f} & \frac{-a_1\delta\beta+\alpha(a_2\delta+a_4\varepsilon)}{f} & a_3\delta+a_5\varepsilon \\ -\frac{a_6\delta}{f} & \frac{a_6\alpha}{f} & a_7 \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & a_6 & a_7 \end{pmatrix} \in H_{n,0} \right\}.$$

Intersection with $H_{n',m'}$: A matrix in this intersection has polynomial entries. For degree reasons we obtain $a_6 = 0$ and f divides $a_1\alpha\varepsilon - a_2\alpha\delta - a_4\delta\beta = a_1 f - a_2\alpha\delta - (a_4-a_1)\delta\beta$, where the equation holds because of the determinant of g. Since f divides $a_1 f$ it follows that f has to divide $a_2\alpha\delta + (a_1-a_4)\delta\beta = \delta(a_2\alpha + (a_1-a_4)\beta)$. For degree reasons this yields $a_2\alpha + (a_1-a_4)\beta = 0$. Since

f is irreducible the polynomials α and β are coprime. Moreover, at least one of the polynomials α and β has its maximal possible degree. These are the reasons, why we necessarily have $a_2 = 0$ and $a_1 = a_4$. For the intersection we conclude

$$gH_{n,0}g^{-1} \cap H_{n',m'} =$$
$$\left\{ \begin{pmatrix} a_1 & 0 & a_3\alpha + a_5\beta \\ 0 & a_1 & a_3\delta + a_5\varepsilon \\ 0 & 0 & a_7 \end{pmatrix} \mid a_1, a_7 \in k^\times,\ a_5 \in k,\ a_3 \in k[t] \text{ with } deg(a_3) \leq n \right\}.$$

We deduce that $|gH_{n,0}g^{-1} \cap H_{n',m'}| = (q-1)q^{n+2}$. With $d = n' - n + m'$ it is $\kappa = \frac{d-2n-n'+m-m'}{3} = -n$. By Lemma 3.5.13 we obtain $|\Upsilon_{n',m',n,0}| = (q-1)(q^2-1)^2 q^{2n'+2m'-n+2}$.

According to 2.6.8 this leads to

$$\frac{|\Upsilon_{n',m',n,0}|}{|H_{n',m'}||H_{n,0}|}|gH_{n,0}g^{-1} \cap H_{n',m'}| = \frac{(q-1)(q^2-1)^2 q^{2n'+2m'-n+2}}{(q-1)^2 q^{2n'+3}(q^2-1)(q^2-q)q^{2n+2}}(q-1)q^{n+2} =$$
$$(q+1)q^{2m'-2n-2} = (q+1)q^{d-n'+m'-n-2}$$

double cosets, if $d \geq n' - m' + n + 2 \geq 4$ (cf. 1(i)ii and 2(i)ii).

Now consider the subcase $d > n' - n + 2m + m'$. Then the degree restraints are given by

$$deg(\alpha) \leq \tfrac{d+2n'-2n-m'}{3},\ deg(\beta) \leq \tfrac{d+2n'+n-m'}{3},\ deg(\gamma) \leq \tfrac{d+2n'+n-m'}{3},\ deg(\delta) \leq \tfrac{d-n'-2n+2m'}{3},\ deg(\varepsilon) \leq \tfrac{d-n'+n+2m'}{3},\ deg(\theta) \leq \tfrac{d-n'+n+2m'}{3},\ deg(\rho) \leq \tfrac{d-n'+n-m'}{3},\ deg(\iota) \leq \tfrac{d-n'+n-m'}{3}.$$

Moreover, we still have $d > 2n + n' - m - 2m'$, which means $\kappa + m' > 0$. Furthermore we know that δ has to be non-zero, because otherwise $det(M) = \alpha(\varepsilon\iota - \rho\theta) = \lambda f$ yields a contradiction to the Irreducibility of f, since $deg(\varepsilon\iota - \rho\theta) \leq \frac{2d-2n'+2n+m'}{3} < d-n'+m' < d$ and $deg(\alpha) \leq \frac{d+2n'-2n-m'}{3} < d-m' < d$.

If we multiply with matrices from $H_{n',m'}$ and $H_{n,0}$, the polynomial δ is just multiplied with some non-zero elements from the field k, i.e. the degree of δ does not change by this multiplication. This leads to the following case distinction when we want to choose a suitable representative for the double coset:

- If $deg(\delta)$ is equal to the maximal possible degree for δ then we choose $g = \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ 0 & \rho & \iota \end{pmatrix}$, such that δ, ι and β have their respective maximal possible degree and all other entries have not their respective maximal possible degree, as a representative for the double coset.

- If δ has not its maximal possible degree, we choose as representative the matrix $g = \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ 0 & \rho & \iota \end{pmatrix}$, with $\alpha,\ \varepsilon$ and ι of their respective maximal possible degree and all entries different from these three have not their respective maximal possible degree.

Thus we have

$$H_{n',m'}g = \left\{ \begin{pmatrix} a_1\alpha + a_2\delta & a_1\beta + a_2\varepsilon + a_3\rho & a_1\gamma + a_2\theta + a_3\iota \\ a_4\delta & a_4\varepsilon + a_5\rho & a_4\theta + a_5\iota \\ 0 & a_6\rho & a_6\iota \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{pmatrix} \in H_{n',m'} \right\}$$

and

$$gH_{n,0} = \left\{ \begin{pmatrix} b_1\alpha & b_2\alpha + b_4\beta + b_6\gamma & b_3\alpha + b_5\beta + b_7\gamma \\ b_1\delta & b_2\delta + b_4\varepsilon + b_6\theta & b_3\delta + b_5\varepsilon + b_7\theta \\ 0 & b_4\rho + b_6\iota & b_5\rho + b_7\iota \end{pmatrix} \mid \begin{pmatrix} b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 \\ 0 & b_6 & b_7 \end{pmatrix} \in H_{n,0} \right\}.$$

The intersection of these two sets is determined by the equations:

$$a_4\delta = b_1\delta,\ a_1\alpha + a_2\delta = b_1\alpha,\ a_6\rho = b_4\rho + b_6\iota,$$

$$a_4\varepsilon + a_5\rho = b_2\delta + b_4\varepsilon + b_6\theta,\ a_1\beta + a_2\varepsilon + a_3\rho = b_2\alpha + b_4\beta + b_6\gamma,\ a_6\iota = b_5\rho + b_7\iota,$$

$$a_4\theta + a_5\iota = b_3\delta + b_5\varepsilon + b_7\theta,\ a_1\gamma + a_2\theta + a_3\iota = b_3\alpha + b_5\beta + b_7\gamma.$$

With $\delta \neq 0$ it follows $a_4 = b_1$. Since ι is of degree $\frac{d-n'+n-m'}{3} > 0$ and the polynomials ι and ρ are coprime we deduce from $a_6\rho = b_4\rho + b_6\iota$ that $b_6 = 0$ and $a_6 = b_4$. The equation $a_6\iota = b_5\rho + b_7\iota$ implies $a_6 = b_7$ and $b_5 = 0$.

Next we distinguish the two cases for the representative we have chosen.

- If δ has its maximal possible degree, then $a_1\alpha + a_2\delta = b_1\alpha$ yields $b_1 = a_1$ and $a_2 = 0$, since α and δ are coprime polynomials and $deg(\delta) = \kappa + m' > 0$. Using $a_1\beta + a_2\varepsilon + a_3\rho = b_2\alpha + b_4\beta + b_6\gamma$ we get $a_1 = b_4$ and hence $a_3\rho = b_2\alpha$. The other equations are $a_5\rho = b_2\delta,\ a_5\iota = b_3\delta$ and $a_3\iota = b_3\alpha$. Since α and δ are coprime polynomials we know from the latter two equations that there exists a polynomial $b \in k[t]$, such that $b_3 = b\iota$. Moreover, the polynomial b has degree $deg(b) = deg(b_3) - deg(\iota) \leq n - (\kappa + n) = -\kappa$. With the other equations we arrive at $b_2 = b\rho,\ a_5 = b\delta$ and $a_3 = b\alpha$. Thus $H_{n',m'}g \cap gH_{n,0} = \left\{ \begin{pmatrix} a_1 & 0 & b\alpha \\ 0 & a_1 & b\delta \\ 0 & 0 & a_1 \end{pmatrix} g \mid a_1 \in k^\times,\ deg(b) \leq -\kappa \right\}$.
- If δ has not its maximal possible degree we deduce again from $a_1\alpha + a_2\delta = b_1\alpha$ that $a_1 = b_1$ and $a_2 = 0$, but now we use the fact that $deg(\alpha) = \kappa + n' > 0$ and the polynomials α and δ are coprime. The equation $a_4\varepsilon + a_5\rho = b_2\delta + b_4\varepsilon + b_6\theta$ leads to $a_4 = b_4$ and whence $a_5\rho = b_2\delta$. The

other equations are $a_5\iota = b_3\delta,\ a_3\iota = b_3\alpha$ and $a_3\rho = b_2\alpha$. Now it follows similar to the previous case that there exists some $b \in k[t]$ with $deg(b) \leq -\kappa$ and $a_3 = b\alpha,\ a_5 = b\delta,\ b_2 = b\rho$ and $b_3 = b\iota$. Therefore we have again

$$H_{n',m'}g \cap gH_{n,0} = \left\{ \begin{pmatrix} a_1 & 0 & b\alpha \\ 0 & a_1 & b\delta \\ 0 & 0 & a_1 \end{pmatrix} g \mid a_1 \in k^\times,\ deg(b) \leq -\kappa \right\}.$$

This implies that we have only one possible length for the double cosets in this case. In particular, $|H_{n',m'}g \cap gH_{n,0}| = q^{-\kappa+1}$ and hence $|H_{n',m'}gH_{n,0}| = (q-1)^2 q^{2n'+3}(q^2-1)(q^2-q)q^{2n+2}q^{\kappa-1}$.

Remember that $-\kappa < m'$. Due to Lemma 3.5.12 we have $|\Upsilon_{n',m',n,0}| = (q-1)^2(q^2-1)^2 q^{d+2\kappa+2n+n'+m'+1}$. Whence, we get

$$\frac{|\Upsilon_{n',m',n,0}|}{|H_{n',m'}gH_{n,0}|} = \frac{(q-1)^2(q^2-1)^2 q^{d+2\kappa+2n+n'+m'+1}}{(q-1)^2 q^{2n'+3}(q^2-1)(q^2-q)q^{2n+2}q^{\kappa-1}} = (q+1)q^{d+\kappa-n'+m'-4}$$

double cosets, if $d \geq n' - m' - \kappa + 4 \geq 1 + 1 + 4 = 6$ (cf. 1(p)i and 2(p)i).

3. $n > m > 0$

   a) $n' > m' = 0$: According to the symmetry 3.7.4 we compute this case from the solutions of the case $n = m > 0$ and $n' > m' > 0$. Therefore we find one double coset in the case $d = n - n' = m$ (cf. 1(a)ii and 2(a)ii). Moreover, we obtain q^{d-m-2} double cosets, if we have $d > n - n' > m$, together with $d \geq m + 2 \geq 3$ (cf. 1(b)i and 2(b)i).

   b) $n' = m' > 0$: Here it is again possible to use the symmetry 3.7.4. Due to the solutions for the case $n > m = 0$ and $n' > m' > 0$ we conclude the following solutions for the given case: We work with the case distinction into the cases $d < n' + m' - n + 2m,\ d = n' + m' - n + 2m$ and $d > n' + m' - n + 2m$.

      In the first case $d < n'+m'-n+2m$ we have one double coset if $d = n-m$ and $n = n'$ holds (cf. 1(a)ii and 2(a)ii). Furthermore there are $q^{d-n+m-2}$ double cosets for $n' > n$ and if we have additionally the equation $d = 2n' - n - m \geq 3$ for the degree (cf. 1(b)ii and 2(b)ii).

      If we consider the second case $d = n' + m' - n + 2m$, the symmetry argument yields the following solutions: For $d = n = n' + m$ there exists only one double coset (cf. 1(a)iii and 2(a)iii). In the case $n' > n - m$ we find q^{d-n-2} double cosets, if the degree fulfills $d \geq n + 2 \geq 4$ (cf. 1(c)i and 2(c)i).

      To solve the last case $d > n' + m' - n + 2m$ with the symmetry argument we do again case distinction: The case $d \leq 2n - m - n'$ implies $(q+1)q^{d-n'-m-2}$ double cosets, if $d = 2n - m - n',\ n' < n - m$ and $d \geq n' + m + 2 \geq 4$ holds (cf. 1(i)i and 2(i)i). In the last subcase $d > 2n - m - n'$ we obtain $(q+1)q^{d+\kappa-m-4}$ double cosets, if $d \geq m + 4 - \kappa \geq 1 + 4 + 1 = 6$ (cf. 1(o)i and 2(o)i).

c) $n' > m' > 0$: We divide this case into the three subcases $d < 2n + n' - m - 2m'$, $d = 2n + n' - m - 2m'$ and $d > 2n + n' - m - 2m'$.

We start with $d < 2n+n'-m-2m'$. Whence, $\delta = 0$ and $det(M) = \alpha(\varepsilon\iota-\rho\theta) = \lambda f$. The Irreducibility of f yields two cases:

First case: $\alpha = \mu f$, for $\mu \in k^\times$, and $\varepsilon\iota - \rho\theta \in k^\times$. This implies $d = deg(\alpha) \leq \frac{d+2n'-2n+m-m'}{3}$, whence $2d \leq 2n' - 2n + m - m'$. According to $deg(\varepsilon\iota - \rho\theta) \leq \frac{2d-2n'+2n-m+m'}{3}$ and $\varepsilon\iota - \rho\theta \in k^\times$ it follows $2d = 2n' - 2n + m - m'$. This leads to $deg(\rho) \leq -\frac{m+m'}{2} < 0$, $deg(\varepsilon) \leq \frac{m'-m}{2}$ and $deg(\iota) \leq \frac{m-m'}{2}$. Since $\varepsilon\iota$ is an element in $k^\times$, these degree restraints imply $m = m'$. Then $d = n' - n$. The other degree restraints are $deg(\alpha) = d$, $deg(\beta) \leq n' - m$, $deg(\gamma) \leq n'$, $deg(\theta) \leq m$.

We calculate

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \varepsilon & \theta \\ 0 & 0 & \iota \end{pmatrix} = \begin{pmatrix} \mu & \beta & \gamma \\ 0 & \varepsilon & \theta \\ 0 & 0 & \iota \end{pmatrix} \begin{pmatrix} f & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in$$
$$H_{n',m'} \begin{pmatrix} f & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} H_{n,m}.$$

Hence we obtain one double coset in this case (cf. 1(a)i and 2(a)i).

Second case: $\alpha \in k^\times$ and $\varepsilon\iota-\rho\theta = \mu f$, for some $\mu \in k^\times$. The degree restraints imply $d = deg(\varepsilon\iota - \rho\theta) \leq \frac{2d-2n'+2n-m+m'}{3}$ and hence $d \leq 2n - 2n' - m + m'$. Now $\alpha \in k^\times$ yields $d = 2n - 2n' - m + m'$. We obtain the following degree restraints $deg(\beta) \leq n - m$, $deg(\gamma) \leq n$, $deg(\varepsilon) \leq n - n' - m + m'$, $deg(\theta) \leq n - n' - m$, $deg(\rho) \leq n - n' - m$, $deg(\iota) \leq n - n'$. So we have $n \geq n'$, since otherwise $\vartheta = 0 = \rho = \iota$.

Suppose $n = n'$. In this case $deg(\rho) < 0$ and hence $\mu f = \varepsilon\iota$. According to the degree restraints and the Irreducibility of f we know $\iota = \mu \in k^\times$ and $\varepsilon = f$. We conclude

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \varepsilon & \theta \\ 0 & 0 & \iota \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta \\ 0 & 0 & \iota \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in$$
$$H_{n',m'} \begin{pmatrix} 1 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix} H_{n,m}.$$

In particular, there is only one double coset in this case (cf. 1(a)i and 2(a)i).

Consider the case $n > n'$. We distinguish three subcases: $n' + m > n$, $n' + m = n$ and $n' + m < n$.

For $n'+m>n$ we conclude $\rho=0$ and $\varepsilon\iota=\mu f$ from the degree restraints. Moreover, $deg(\varepsilon)\le d-n+n'<d$ and the Irreducibility of f imply $\varepsilon=\mu\in k^\times$ and $\iota=f$. This means $2n-2n'-m+m'=d=deg(\iota)\le n-n'$, which leads to $n+m'\le n'+m$. Assume $n+m'<n'+m$, then $0=deg(\varepsilon)\le n-n'-m+m'<0$ is a contradiction. Hence, $n+m'=n'+m$ and $d=n-n'$.

We obtain

$$\begin{pmatrix}\alpha&\beta&\gamma\\0&\varepsilon&\theta\\0&0&\iota\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&f\end{pmatrix}\begin{pmatrix}\alpha&\beta&\gamma\\0&\varepsilon&\theta\\0&0&1\end{pmatrix}\in H_{n',m'}\begin{pmatrix}1&0&0\\0&1&0\\0&0&f\end{pmatrix}H_{n,m}.$$

Which says there is only one double coset (cf. 1(a)ii and 2(a)ii).

If $n'+m=n$, then $deg(\rho)\le 0$ and $d=n-n'+m'$. Assume $\rho=0$. This implies with $\varepsilon\iota=f$ a contradiction to the irreducibility of f, since $deg(\varepsilon)\le n-n'-m+m'<d$ and $deg(\iota)<d$. We obtain $\rho\in k^\times$ and from the determinant of M we deduce $\theta=\rho^{-1}(\varepsilon\iota-f)$. Whence,

$$\begin{pmatrix}\alpha&\beta&\gamma\\0&\varepsilon&\theta\\0&\rho&\iota\end{pmatrix}=\begin{pmatrix}1&0&0\\0&-\rho^{-1}&\rho^{-1}\varepsilon\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&f\\0&1&0\end{pmatrix}\begin{pmatrix}\alpha&\beta&\gamma\\0&\rho&\iota\\0&0&1\end{pmatrix}\in H_{n',m'}\begin{pmatrix}1&0&0\\0&0&f\\0&1&0\end{pmatrix}H_{n,m}.$$

So there is only one double coset (cf. 1(a)iii and 2(a)iii).

Suppose $n'+m<n$. Via multiplication by a suitable matrix from $H_{n,m}$ we can assume $\alpha=1,\ \beta=0=\gamma$. In particular, we choose $g:=\begin{pmatrix}1&0&0\\0&\varepsilon&\theta\\0&\rho&\iota\end{pmatrix}$ with $det(g)=f$ as a representative for the double coset of M. Then we compute

$$g^{-1}H_{n',m'}g=$$
$$\left\{\begin{pmatrix}1&0&0\\0&\frac{\iota}{f}&-\frac{\theta}{f}\\0&-\frac{\rho}{f}&\frac{\varepsilon}{f}\end{pmatrix}\begin{pmatrix}a_1&a_2&a_3\\0&a_4&a_5\\0&0&a_6\end{pmatrix}\begin{pmatrix}1&0&0\\0&\varepsilon&\theta\\0&\rho&\iota\end{pmatrix}\mid\begin{pmatrix}a_1&a_2&a_3\\0&a_4&a_5\\0&0&a_6\end{pmatrix}\in H_{n',m'}\right\}=$$
$$\left\{\begin{pmatrix}a_1&a_2\varepsilon+a_3\rho&a_2\theta+a_3\iota\\0&\frac{a_4\iota\varepsilon+a_5\iota\rho-a_6\rho\theta}{f}&\frac{a_4\iota\theta+a_5\iota^2-a_6\iota\theta}{f}\\0&\frac{-a_4\rho\varepsilon-a_5\rho^2+a_6\rho\varepsilon}{f}&\frac{-a_4\rho\theta-a_5\iota\rho+a_6\varepsilon\iota}{f}\end{pmatrix}\mid\begin{pmatrix}a_1&a_2&a_3\\0&a_4&a_5\\0&0&a_6\end{pmatrix}\in H_{n',m'}\right\}.$$

If we take the intersection with $H_{n,m}$ the entries of a matrix are polynomials. So we obtain $(a_4-a_6)\varepsilon+a_5\rho=0$. Since ρ and ε are coprime polynomials and either ρ or ε is of its maximal possible degree we deduce $a_4=a_6$ and $a_5=0$.

Hence

$$g^{-1}H_{n',m'}g\cap H_{n,m}=$$
$$\left\{\begin{pmatrix} a_1 & a_2\varepsilon+a_3\rho & a_2\theta+a_3\iota \\ 0 & a_4 & 0 \\ 0 & 0 & a_4 \end{pmatrix} \mid a_1,a_4\in k^\times, deg(a_2)\le n'-m', deg(a_3)\le n'\right\}.$$

Thus $|g^{-1}H_{n',m'}g\cap H_{n,m}|=(q-1)q^{2n'-m'+2}$.

Using Lemma 3.5.9 and Remark 2.6.8 we obtain

$$\frac{|\Upsilon_{n',m',n,m}|}{|H_{n',m'}||H_{n,m}|}|g^{-1}H_{n',m'}g\cap H_{n,m}|=$$
$$\frac{(q-1)^2(q^d+q^{d-1})(q^2-q)q^{2n-m+2}}{(q-1)^2q^{2n'+3}(q-1)^2q^{2n+3}}(q-1)q^{2n'-m'+2}=(q+1)q^{d-m-m'-2}$$

double cosets, if the degree of f is at least $d\ge m+m'+2\ge 4$ (cf. 1(j)i and 2(j)i).

Consider the case $d=2n+n'-m-2m'$. Assume $\delta=0$. As in the case $d<2n+n'-m-2m'$ above we have $d=n'-n$ and $m=m'$ or $d=2n-2n'-m+m'$. The first case yields $n'-n=d=2n+n'-m-2m'=2n+n'-3m$ and hence $n=m$, which is a contradiction. In the second case $2n-2n'-m+m'=d=2n+n'-m-2m'$, which means $n'=m'$, again a contradiction. Therefore, we conclude $\delta\in k^\times$. We have

$deg(\alpha)\le n'-m'$, $deg(\beta)\le n+n'-m-m'$, $deg(\gamma)\le n+n'-m'$, $deg(\delta)=0$, $deg(\varepsilon)\le n-m$, $deg(\theta)\le n$, $deg(\rho)\le n-m-m'$, $deg(\iota)\le n-m'$

for the degree restraints. Now we see that $n\ge m'$, because otherwise $\vartheta=0=\rho=\iota$ contradicts to M is invertible.

Next we distinguish several subcases.

Suppose $n=m'$, then $d=n'-m$, $\rho=0$, $\iota\in k^\times$ and $\alpha\varepsilon-\delta\beta=f$. We may write $\beta=\delta^{-1}(\alpha\varepsilon-f)$ and compute

$$\begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ 0 & 0 & \iota \end{pmatrix}=\begin{pmatrix} \delta^{-1} & \delta^{-1}\alpha & \gamma \\ 0 & 1 & \theta \\ 0 & 0 & \iota \end{pmatrix}\begin{pmatrix} 0 & -f & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \delta & \varepsilon & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\in$$
$$H_{n',m'}\begin{pmatrix} 0 & -f & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}H_{n,m}.$$

This means there is only one double coset in this case (cf. 1(a)i and 2(a)i).

Assume $m' < n < m + m'$. Then we obtain $\rho = 0$ and $det(M) = \iota(\alpha\varepsilon - \delta\beta) = \lambda f$, but we also have the degree restraints $deg(\iota) \leq n - m' < d$ and $deg(\alpha\varepsilon - \delta\beta) \leq n + n' - m - m' < d$, which is a contradiction to the Irreducibility of f.

If $n = m + m'$, we have $d = n' + m$ and ρ is an element in the field k. The Assumption $\rho = 0$ yields a contradiction by similar arguments to the above case $m' < n < m + m'$. So we conclude $\rho \neq 0$. From $det(M) = \iota(\alpha\varepsilon - \delta\beta) - \rho(\alpha\theta - \delta\gamma) = \lambda f$ we derive $\gamma = \delta^{-1}(\alpha\theta + \rho^{-1}(\lambda f - \iota(\alpha\varepsilon - \delta\beta)))$. Using this expression for γ, we calculate

$$\begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix} =$$

$$\begin{pmatrix} \alpha & \delta^{-1}\alpha\varepsilon + \delta^{-1}(\delta\beta - \alpha\varepsilon) & \delta^{-1}(\alpha\theta + \rho^{-1}(\lambda f - \iota(\alpha\varepsilon - \delta\beta))) \\ \delta & \varepsilon & \theta \\ 0 & \rho & \iota \end{pmatrix} =$$

$$\begin{pmatrix} \delta^{-1}\rho^{-1} & \delta^{-1}\alpha & \delta^{-1}\rho^{-1}(\delta\beta - \alpha\varepsilon) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & f \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \delta & \varepsilon & \theta \\ 0 & \rho & \iota \\ 0 & 0 & \lambda \end{pmatrix} \in$$

$$H_{n',m'} \begin{pmatrix} 0 & 0 & f \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} H_{n,m}.$$

And hence there is only one double coset (cf. 1(a)iv and 2(a)iv).

Suppose $n > m+m'$. Since δ is a non-zero element in the field k we can choose as representative for the double coset of M the matrix $g := \begin{pmatrix} 0 & \beta & \gamma \\ 1 & 0 & 0 \\ 0 & \rho & \iota \end{pmatrix}$ with $det(g) = f$. Using this representative we compute

$$g^{-1}H_{n',m'}g =$$

$$\left\{ \begin{pmatrix} 0 & 1 & 0 \\ -\frac{\iota}{f} & 0 & -\frac{\gamma}{f} \\ \frac{\rho}{f} & 0 & \frac{\beta}{f} \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{pmatrix} \begin{pmatrix} 0 & \beta & \gamma \\ 1 & 0 & 0 \\ 0 & \rho & \iota \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{pmatrix} \in H_{n',m'} \right\} =$$

$$\left\{ \begin{pmatrix} a_4 & a_5\rho & a_5\iota \\ \frac{-a_2\iota}{f} & \frac{-\iota(a_1\beta + a_3\rho) + a_6\rho\gamma}{f} & \frac{-\iota(a_1\gamma + a_3\iota) + a_6\iota\gamma}{f} \\ \frac{a_2\rho}{f} & \frac{\rho(a_1\beta + a_3\rho) - a_6\rho\beta}{f} & \frac{\rho(a_1\gamma + a_3\iota) - a_6\iota\beta}{f} \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{pmatrix} \in H_{n',m'} \right\}.$$

Consider the intersection $g^{-1}H_{n',m'}g \cap H_{n,m}$: Since the entries of a matrix in this intersection are polynomials we deduce $a_2 = 0$ and $\rho((a_1 - a_6)\beta + a_3\rho) = 0$.

If we assume $\rho = 0$, then it follows similar to the case $m' < n < m+m'$ above, that $det(g) = -\iota\beta = f$, but $deg(\iota) \leq n - m' = d - n - n' + m - m' < d$ and $deg(\beta) \leq n+n'-m-m' = d-n-m' < d$ is a contradiction to the Irreducibility of f. Therefore, we conclude $(a_1 - a_6)\beta + a_3\rho = 0$. Since β and ρ are coprime polynomials with at least one of them of its maximal possible degree, we get $a_1 = a_6$ and $a_3 = 0$. This leads to

$$g^{-1}H_{n',m'}g \cap H_{n,m} =$$
$$\left\{ \begin{pmatrix} a_4 & a_5\rho & a_5\iota \\ 0 & a_1 & 0 \\ 0 & 0 & a_1 \end{pmatrix} \mid a_1, a_4 \in k^\times,\ a_5 \in k[t] \text{ with } deg(a_5) \leq m' \right\}$$

and hence $|g^{-1}H_{n',m'}g \cap H_{n,m}| = (q-1)q^{m'+1}$.

Now $d = 2n + n' - m - 2m'$ implies $\kappa = -m' < 0$. By Lemma 3.5.12 1.b) the cardinality of the set $\Upsilon_{n',m',n,m}$ is equal to $|\Upsilon_{n',m',n,m}| = (q-1)^2(q^2 - 1)q^{d+\kappa+2n+n'-m+3}$. Furthermore, 2.6.8 gives us $|H_{n,m}| = (q-1)^2 q^{2n+3}$. In total we obtain

$$\frac{|\Upsilon_{n',m',n,m}|}{|H_{n',m'}||H_{n,m}|}|g^{-1}H_{n',m'}g \cap H_{n,m}| = \frac{(q-1)^2(q^2-1)q^{d+\kappa+2n+n'-m+3}}{(q-1)^2q^{2n'+3}(q-1)^2q^{2n+3}}(q-1)q^{m'+1} = (q+1)q^{d-m-n'-2}$$

double cosets, if the degree $d \geq m + n' + 2 \geq 5$ is large enough (cf. 1(j)i and 2(j)i).

Next we consider the case $d > 2n + n' - m - 2m'$. In order to do that, we distinguish three subcases, in particular the cases $d < n' - n + 2m + m'$, $d = n' - n + 2m + m'$ and $d > n' - n + 2m + m'$.

Suppose $d < n' - n + 2m + m'$. In order to solve this case we can use the symmetry 3.7.4 and the solutions from the above subcase $d < 2n+n'-m-2m'$. But since we have additionally the condition $d > 2n + n' - m - 2m'$ most of the solutions we computed in the case above can not occur in the case we consider now. First we consider the case $d = n - n' = m - m'$. This implies $m-m' = d > 2n+n'-m-2m'$ and hence we have with $0 > 2(n-m)+n'-m'$ a contradiction. So this is one of the cases which does not occur. Next we computed some double cosets in the case $d = n' - n + m' - m$. With our additional condition $n' - n + m' - m = d > 2n + n' - m - 2m'$ we deduce $m' > n$. This means that we calculated some subcases in the other case which can not occur here. Only for the case $m' > n$ and $d = n' - n + m' - m$ we obtain $(q+1)q^{d-n'+m'-n+m-2}$ double cosets, if $d \geq n' - m' + n - m + 2 \geq 4$ (cf. 1(l)i and 2(l)i).

Consider the case $d = n' - n + 2m + m'$. We use again the symmetry 3.7.4 and the solutions for the subcase $d = 2n+n'-m-2m'$. As in the previous case we have the additional condition $d > 2n + n' - m - 2m'$, which implies that some of the possible cases can not occur now. If we assume $d = n - n' + m'$ and $n' = n - m$, then we conclude from $n - n' + m' = d > 2n + n' - m - 2m'$ that $3m' > n - m + 2n' = 3n'$, i.e. with $m' > n'$ a contradiction. Therefore this case does not occur. Next $d = n + n' - m'$ and $n - m = m'$ could be possible. But this yields $n + n' - m' = d > 2n + n' - m - 2m'$ and whence $m' > n - m$. We conclude that the only case which occurs is the case $m' > n - m$. By symmetry we have $(q+1)q^{d-n'+m'-n-2}$ double cosets in this case, if we also have that $d \geq n' - m' + n + 2 \geq 1 + 2 + 2 = 5$ (cf. 1(k)i and 2(k)i).

Let $d > n' - n + 2m + m'$. So we know $\kappa + m' > 0$ and $\kappa + n - m > 0$. Remember the degree restraints for the entries of the matrix M:

$$deg(\alpha) \leq \kappa + n',\ deg(\beta) \leq \kappa + n - m + n',\ deg(\gamma) \leq \kappa + n + n',\ deg(\delta) \leq \kappa + m',\ deg(\varepsilon) \leq \kappa + n - m + m',\ deg(\theta) \leq \kappa + n + m',\ deg(\rho) \leq \kappa + n - m,\ deg(\iota) \leq \kappa + n.$$

By multiplication with matrices from $H_{n',m'}$ and $H_{n,m}$ we multiply the polynomials δ and ρ just with some non-zero elements in the field k, which implies that the degree of δ and ρ can not be changed via this multiplication. This observation leads to the following cases:

- If both, δ and ρ have not their respective maximal possible degree we choose as representative for the double coset the matrix $g = \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ 0 & \rho & \iota \end{pmatrix}$, where the entries on the diagonal, α, ε and ι, have their respective maximal possible degree and all the entries not on the diagonal have less than the possible degree.
- If δ has its maximal possible degree and ρ has not its maximal possible degree, we choose the representative $g = \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ 0 & \rho & \iota \end{pmatrix}$ with δ, β and ι of their respective maximal possible degree and all other entries have not their respective maximal possible degree.
- If δ is not of its maximal possible degree, but ρ is of its maximal possible degree, we take $g = \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ 0 & \rho & \iota \end{pmatrix}$ with $deg(\rho)$, $deg(\alpha)$ and $\deg(\theta)$ respectively maximal and all other entries have less degree than it is possible, as representative for the double coset.
- If δ and ρ are both of their respective maximal possible degree, then we

use the representative $g = \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ 0 & \rho & \iota \end{pmatrix}$, where δ, ρ and γ are of their respective maximal possible degree and all the other entries have not their respective maximal possible degree.

Next we intersect the two sets

$$H_{n',m'}g = \left\{ \begin{pmatrix} a_1\alpha + a_2\delta & a_1\beta + a_2\varepsilon + a_3\rho & a_1\gamma + a_2\theta + a_3\iota \\ a_4\delta & a_4\varepsilon + a_5\rho & a_4\theta + a_5\iota \\ 0 & a_6\rho & a_6\iota \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{pmatrix} \in H_{n',m'} \right\}$$

and

$$gH_{n,m} = \left\{ \begin{pmatrix} b_1\alpha & b_2\alpha + b_4\beta & b_3\alpha + b_5\beta + b_6\gamma \\ b_1\delta & b_2\delta + b_4\varepsilon & b_3\delta + b_5\varepsilon + b_6\theta \\ 0 & b_4\rho & b_5\rho + b_6\iota \end{pmatrix} \mid \begin{pmatrix} b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 \\ 0 & 0 & b_6 \end{pmatrix} \in H_{n,m} \right\}.$$

Since $\delta \neq 0 \neq \rho$ we deduce from $b_1\delta = a_4\delta$ that $a_4 = b_1$ and from $a_6\rho = b_4\rho$ we get $a_6 = b_4$. The equation $(a_1 - b_1)\alpha + a_2\delta = 0$ yields $a_1 = b_1$ and $a_2 = 0$, because α and δ are coprime polynomials and if the degree of δ is not maximal we have chosen g such that α has maximal possible degree. Analogous we conclude $a_6 = b_6$ and $b_5 = 0$ from the equation $(a_6 - b_6)\iota = b_5\rho$, since ρ and ι are coprime polynomials and if we have that the degree of ρ is not maximal, we know that ι has maximal possible degree. We consider the remaining equations:

$$(a_4 - b_4)\varepsilon + a_5\rho = b_2\delta,\ (a_1 - b_4)\beta + a_3\rho = b_2\alpha,$$

$$(a_4 - b_6)\theta + a_5\iota = b_3\delta,\ (a_1 - b_6)\gamma + a_3\iota = b_3\alpha.$$

Now we can conclude that $a_1 = a_4 = a_6 = b_1 = b_4 = b_6$. To see this we consider the four subcases:

- If δ and ρ have not their respective maximal possible degree the degree of ε is maximal and the first equation yields $a_1 = a_4 = a_6 = b_1 = b_4 = b_6$.
- For δ of its maximal possible degree and ρ not of its maximal possible degree we know that β has its maximal possible degree and the second equation implies $a_1 = a_4 = a_6 = b_1 = b_4 = b_6$.
- The case δ has not its maximal possible degree, but ρ has its maximal possible degree yields that θ is of its maximal possible degree and hence by the third equation we deduce $a_1 = a_4 = a_6 = b_1 = b_4 = b_6$.
- In the last case δ and ρ have their respective maximal possible degree and also γ is of its maximal possible degree. Then the fourth equation gives us $a_1 = a_4 = a_6 = b_1 = b_4 = b_6$.

So we obtain in all four cases that the remaining equations are given by

$$a_5\rho = b_2\delta,\ a_3\rho = b_2\alpha,\ a_5\iota = b_3\delta,\ a_3\iota = b_3\alpha.$$

Whence there exists a polynomial $b \in k[t]$, such that $deg(b) \leq -\kappa$ and $a_3 = b\alpha,\ a_5 = b\delta,\ b_3 = b\iota$ and $b_2 = b\rho$. Therefore there exists only one possible size for the intersection, in particular,

$$H_{n',m'}g \cap gH_{n,m} = \left\{ \begin{pmatrix} a_1 & 0 & b\alpha \\ 0 & a_1 & b\delta \\ 0 & 0 & a_1 \end{pmatrix} g \mid a_1 \in k^\times,\ b \in k[t] \text{ with } deg(b) \leq -\kappa \right\},$$

which means $|H_{n',m'}g \cap gH_{n,m}| = q^{-\kappa+1}$. For this reason there is only one possible length for the double cosets, i.e. $|H_{n',m'}gH_{n,m}| = (q-1)^4 q^{2n'+2n+4+\kappa-1}$. By Lemma 3.5.13 we obtain $|\Upsilon_{n',m',n,m}| = (q-1)^2(q^2-1)^2 q^{d+2\kappa+2n-m+n'+m'+1}$. So we have

$$\frac{|\Upsilon_{n',m',n,m}|}{|H_{n',m'}gH_{n,m}|} = \frac{(q-1)^2(q^2-1)^2 q^{d+2\kappa+2n-m+n'+m'+1}}{(q-1)^4 q^{2n'+2n+4+\kappa-1}} = (q+1)^2 q^{d+\kappa-m+m'-n'-2}$$

double cosets in this case for $d \geq m-\kappa-m'+n'+2 \geq 1+1+1+2 = 5$ (cf. 1(q)i and 2(q)i).

3.7.2. The case $\kappa = 0$, i.e. $d = 2n+n'-m+m'$

Now we obtain the following degree restraints $deg(\alpha) \leq n',\ deg(\beta) \leq n+n'-m,\ deg(\gamma) \leq n+n',\ deg(\delta) \leq m',\ deg(\varepsilon) \leq n-m+m',\ deg(\theta) \leq n+m',\ deg(\vartheta) \leq 0,\ deg(\rho) \leq n-m,\ deg(\iota) \leq n$. In this case we consider 16 subcases. In particular, we consider four different possibilities for the Indices n and m and in every case we have again four cases for the Indices n' and m'.

1. $n > m > 0$:

   a) $n' = m' = 0$: In this case we have $d = 2n-m$ and the degree restraints of the entries of the matrix M are

   $$deg(\alpha) \leq 0,\ deg(\beta) \leq n-m,\ deg(\gamma) \leq n,\ deg(\delta) \leq 0,\ deg(\varepsilon) \leq n-m,\ deg(\theta) \leq n,\ deg(\vartheta) \leq 0,\ deg(\rho) \leq n-m,\ deg(\iota) \leq n.$$

   By multiplication with a suitable matrix from $H_{0,0}$ we can assume $\alpha = 1$ and $\vartheta = 0 = \delta$. A multiplication with a suitable matrix from $H_{n,m}$ yields $\beta = 0 = \gamma$. Hence, we choose the matrix $g := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & \theta \\ 0 & \rho & \iota \end{pmatrix}$ with $det(g) = f$ as representative for the double coset of M.

   We compute

$$g^{-1}H_{0,0}g =$$
$$\left\{\begin{pmatrix}1 & 0 & 0\\ 0 & \frac{\iota}{f} & -\frac{\theta}{f}\\ 0 & -\frac{\rho}{f} & \frac{\varepsilon}{f}\end{pmatrix}\begin{pmatrix}a_1 & a_2 & a_3\\ a_4 & a_5 & a_6\\ a_7 & a_8 & a_9\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \varepsilon & \theta\\ 0 & \rho & \iota\end{pmatrix} \mid \begin{pmatrix}a_1 & a_2 & a_3\\ a_4 & a_5 & a_6\\ a_7 & a_8 & a_9\end{pmatrix} \in H_{0,0}\right\} =$$
$$\left\{\begin{pmatrix}a_1 & a_2\varepsilon + a_3\rho & a_2\theta + a_3\iota\\ \frac{a_4\iota - a_7\theta}{f} & \frac{\iota(a_5\varepsilon+a_6\rho)-\theta(a_8\varepsilon+a_9\rho)}{f} & \frac{\iota(a_5\theta+a_6\iota)-\theta(a_8\theta+a_9\iota)}{f}\\ \frac{a_7\varepsilon - a_4\rho}{f} & \frac{-\rho(a_5\varepsilon+a_6\rho)+\varepsilon(a_8\varepsilon+a_9\rho)}{f} & \frac{-\rho(a_5\theta+a_6\iota)+\varepsilon(a_8\theta+a_9\iota}{f}\end{pmatrix} \mid \begin{pmatrix}a_1 & a_2 & a_3\\ a_4 & a_5 & a_6\\ a_7 & a_8 & a_9\end{pmatrix} \in H_{0,0}\right\}.$$

In the intersection $g^{-1}H_{0,0}g \cap H_{n,m}$ the equality $a_4\iota = a_7\theta$ holds. Since ι and θ are coprime polynomials such that at least one of them is of its maximal possible degree, this implies $a_4 = 0 = a_7$. Moreover, we have that $a_8\varepsilon^2 + (a_9 - a_5)\varepsilon\rho - a_6\rho^2 = 0$. Because ρ and ε are coprime polynomials and not both of them are in the filed k, we obtain $a_6 = 0 = a_8$ and $a_5 = a_9$. This leads to

$$g^{-1}H_{0,0}g \cap H_{n,m} =$$
$$\left\{\begin{pmatrix}a_1 & a_2\varepsilon + a_3\rho & a_2\theta + a_3\iota\\ 0 & a_5 & 0\\ 0 & 0 & a_5\end{pmatrix} \mid a_1, a_5 \in k^\times \text{ and } a_2, a_3 \in k\right\},$$

which implies $|g^{-1}H_{0,0}g \cap H_{n,m}| = (q-1)q^2$. According to Lemma 3.5.12 2.c) we get $|\Upsilon_{0,0,n,m}| = (q^3-1)(q-1)(q^2-1)q^{d+2n-m+2} = (q^3-1)(q-1)(q^2-1)q^{2d+2}$. Therefore, we obtain

$$\frac{|\Upsilon_{0,0,n,m}|}{|H_{0,0}||H_{n,m}|}|g^{-1}H_{0,0}g \cap H_{n,m}| = \frac{(q^3-1)(q-1)(q^2-1)q^{2d+2}}{(q^3-1)(q^3-q)q^2(q-1)^2q^{2n+3}}(q-1)q^2 = q^{d-m-2}$$

double cosets in this case, if $d \geq m+2 \geq 3$ (cf. 1(b)i and 2(b)i).

b) $n' > m' = 0$: Now $d = 2n + n' - m$ and

$$deg(\alpha) \leq n',\ deg(\beta) \leq n + n' - m,\ deg(\gamma) \leq n + n',\ deg(\delta) \leq 0,\ deg(\varepsilon) \leq n - m,\ deg(\theta) \leq n,\ deg(\vartheta) \leq 0,\ deg(\rho) \leq n - m,\ deg(\iota) \leq n.$$

From these degree restraints we deduce that we may choose $g := \begin{pmatrix}0 & \beta & \gamma\\ 1 & 0 & 0\\ 0 & \rho & \iota\end{pmatrix}$ with $det(g) = f$ as representative for the double coset of M (if necessary multiply with a suitable matrix from $H_{n',0}$ to get $\alpha = 0 = \vartheta$ and $\delta = 1$ and then use a suitable matrix from $H_{n,m}$ to get $\varepsilon = 0 = \theta$). With this certain representative we calculate

$$g^{-1}H_{n',0}g =$$
$$\left\{\begin{pmatrix}0 & 1 & 0\\ -\frac{\iota}{f} & 0 & -\frac{\gamma}{f}\\ \frac{\rho}{f} & 0 & \frac{\beta}{f}\end{pmatrix}\begin{pmatrix}a_1 & a_2 & a_3\\ 0 & a_4 & a_5\\ 0 & a_6 & a_7\end{pmatrix}\begin{pmatrix}0 & \beta & \gamma\\ 1 & 0 & 0\\ 0 & \rho & \iota\end{pmatrix} \mid \begin{pmatrix}a_1 & a_2 & a_3\\ 0 & a_4 & a_5\\ 0 & a_6 & a_7\end{pmatrix} \in H_{n',0}\right\} =$$

$$\left\{ \begin{pmatrix} a_4 & a_5\rho & a_5\iota \\ \frac{a_6\gamma - a_2\iota}{f} & \frac{a_7\gamma\rho - \iota(a_1\beta + a_3\rho)}{f} & \frac{a_7\gamma\iota - \iota(a_1\gamma + a_3\iota)}{f} \\ \frac{-a_6\beta + a_2\rho}{f} & \frac{-a_7\beta\rho + \rho(a_1\beta + a_3\rho)}{f} & \frac{-a_7\beta\iota + \rho(a_1\gamma + a_3\iota)}{f} \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & a_6 & a_7 \end{pmatrix} \in H_{n',0} \right\}.$$

In the intersection with $H_{n,m}$ we obtain $a_6\gamma = a_2\iota$ and hence $a_6 = 0 = a_2$, since γ and ι are coprime polynomials and $deg(\gamma)$ or $deg(\iota)$ is the maximal possible one. Furthermore, $(a_1 - a_7)\beta + a_3\rho = 0$ implies $a_1 = a_7$ and $a_3 = 0$, because β and ρ are coprime polynomials with the maximal possible degree for either β or ρ. Thus

$$g^{-1}H_{n',0}g \cap H_{n,m} = \left\{ \begin{pmatrix} a_4 & a_5\rho & a_5\iota \\ 0 & a_1 & 0 \\ 0 & 0 & a_1 \end{pmatrix} \mid a_1, a_4 \in k^\times, a_5 \in k \right\}.$$

Therefore, $|g^{-1}H_{n',0}g \cap H_{n,m}| = (q-1)q$. Applying Lemma 3.5.12 2.b) we get $|\Upsilon_{n',0,n,m}| = (q-1)(q^2-1)^2q^{d+2n-m+n'+3} = (q-1)(q^2-1)^2q^{2d+3}$. We calculate

$$\frac{|\Upsilon_{n',0,n,m}|}{|H_{n',0}||H_{n,m}|}|g^{-1}H_{n',0}g \cap H_{n,m}| = \frac{(q-1)(q^2-1)^2q^{2d+3}}{(q^2-1)(q^2-q)q^{2n'+2}(q-1)^2q^{2n+3}}(q-1)q = (q+1)q^{d-n'-m-2}.$$

This implies for $d \geq n' + m + 2 \geq 4$ there are $(q+1)q^{d-n'-m-2}$ double cosets (cf. 1(j)i and 2(j)i).

c) $n' = m' > 0$: Here $d = 2n + 2n' - m \geq 5$ and

$$deg(\alpha) \leq n',\ deg(\beta) \leq n + n' - m,\ deg(\gamma) \leq n + n',\ deg(\delta) \leq n',\ deg(\varepsilon) \leq n - m + n',\ deg(\theta) \leq n + n',\ deg(\vartheta) \leq 0,\ deg(\rho) \leq n - m,\ deg(\iota) \leq n.$$

If we multiply with matrices from $H_{n',n'}$ from left and with matrices from $H_{n,m}$ from right the lower left entry ϑ is only multiplied with some elements in $k^\times$. We deduce that we have two different types of double cosets, in particular, with $\vartheta = 0$ or $\vartheta \neq 0$.

First suppose $\vartheta = 0$:

When we consider again the multiplication with elements from $H_{n',n'}$ from the left and elements from $H_{n,m}$ from the right we find, according to $\vartheta = 0$, that the entry ρ can only be multiplied with non-zero elements of the field. Therefore we have the following two cases:

- If the degree of ρ is not the maximal possible one, then we choose as representative for the double coset the matrix $g := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ 0 & \rho & \iota \end{pmatrix}$, where only the entries on the diagonal have their respective maximal possible degree.

- If ρ is a polynomial of its maximal possible degree, we take the representative $g := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ 0 & \rho & \iota \end{pmatrix}$ with α, ρ and θ of their respective maximal possible degree, while all other entries are polynomials which have less degree than it is possible.

In order to compute the size of the double coset, we calculate first the cardinality of the intersection of

$$H_{n',n'}g = \left\{ \begin{pmatrix} a_1\alpha + a_2\delta & a_1\beta + a_2\varepsilon + a_3\rho & a_1\gamma + a_2\theta + a_3\iota \\ a_4\alpha + a_5\delta & a_4\beta + a_5\varepsilon + a_6\rho & a_4\gamma + a_5\theta + a_6\iota \\ 0 & a_7\rho & a_7\iota \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & a_7 \end{pmatrix} \in H_{n',n'} \right\}$$

with

$$gH_{n,m} = \left\{ \begin{pmatrix} b_1\alpha & b_2\alpha + b_4\beta & b_3\alpha + b_5\beta + b_6\gamma \\ b_1\delta & b_2\delta + b_4\varepsilon & b_3\delta + b_5\varepsilon + b_6\theta \\ 0 & b_4\rho & b_5\rho + b_6\iota \end{pmatrix} \mid \begin{pmatrix} b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 \\ 0 & 0 & b_6 \end{pmatrix} \in H_{n,m} \right\}.$$

In the intersection we have $a_7\rho = b_4\rho$ and hence $a_7 = b_4$ since $\rho \neq 0$. Furthermore there are the equations $(a_1 - b_1)\alpha + a_2\delta = 0 = a_4\alpha + (a_5 - b_1)\delta$. They imply $a_1 = b_1$, $a_2 = 0$ and $a_4 = 0$, $a_5 = b_1$, because in both cases the entry α of the representative g has its maximal possible degree, but the polynomial δ has less degree than it is possible.

- In the case where the degree of ρ is not the maximal possible one we consider the equation $a_7\iota = b_5\rho + b_6\iota$, which yields $a_7 = b_6$ and $b_5 = 0$, since ι is of its maximal possible degree and ρ is not of its maximal possible degree. From the equation $a_4\beta + a_5\varepsilon + a_6\rho = b_2\delta + b_4\varepsilon$ we deduce $(b_1 - b_4)\varepsilon + a_6\rho = b_2\delta$ and hence $b_1 = b_4$, because in this equation only the polynomial ε is of its maximal possible degree. Now the remaining equations are $a_6\rho = b_2\delta$, $a_3\rho = b_2\alpha$, $a_3\iota = b_3\alpha$ and $a_6\iota = b_3\delta$. Since ι and ρ are coprime polynomials and α has degree $n' > 0$ it follows that there exists an element $b \in k$ with $a_3 = b\alpha$, $a_6 = b\delta$ $b_2 = b\rho$ and $b_3 = b\iota$.
- Now consider the case where ρ has its maximal possible degree. Here we start again with the equation $a_7\iota = b_5\rho + b_6\iota$. The degree of ρ is $n - m$ and whence greater than zero. Moreover, the polynomials $\rho \in k[t] \setminus k$ and ι are coprime. Therefore we deduce $a_7 = b_6$ and $b_5 = 0$. With the equation $a_4\gamma + a_5\theta + a_6\iota = b_3\delta + b_5\varepsilon + b_6\theta$ we find $(b_1 - b_6)\theta + a_6\iota = b_3\delta$, which means $b_1 = b_6$ due to the fact that θ has its maximal possible degree and the other two polynomials do not have their respective maximal possible degree. Next we have the equations $a_3\iota = b_3\alpha$, $a_6\iota = b_3\delta$, $a_6\rho = b_2\delta$ and $a_3\rho = b_2\alpha$. Since the polynomials α and δ are coprime and the degree

of ρ is greater than zero, there has to be an element $b \in k$ such that $b_2 = b\rho,\ a_3 = b\alpha,\ a_6 = b\delta$ and $b_3 = b\iota$.

In both cases we found $H_{n',n'}g \cap gH_{n,m} = \left\{ g \begin{pmatrix} a & b\rho & b\iota \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \mid a \in k^\times \text{ and } b \in k \right\}$.

So $|H_{n',n'}g \cap gH_{n,m}| = q$ and hence $|M^{-1}H_{n',n'}M \cap H_{n,m}| = q$ for all possible representatives M for a double coset with lower left entry equal to zero.

By Lemma 3.5.12 2.a) we obtain $|\Upsilon^{\vartheta=0}_{n',n',n,m}| = (q-1)(q^2-1)q^{n'+m'-1}(q-1)(q^2-1)q^{d+2n-m+2} = (q-1)^2(q^2-1)^2q^{2d+1}$. Thus,

$$\frac{|\Upsilon^{\vartheta=0}_{n',n',n,m}|}{|H_{n',n'}||H_{n,m}|} \cdot |M^{-1}H_{n',n'}M \cap H_{n,0}| = \frac{(q-1)^2(q^2-1)^2q^{2d+1}}{(q^2-1)(q^2-q)q^{2n'+2}(q-1)q^{2n+3}} \cdot q = (q+1)q^{d-m-4}.$$

There are $(q+1)q^{d-m-4}$ double cosets for the case with representatives having lower left entry equal to zero.

Now suppose $\vartheta \neq 0$: Using multiplication with a suitable matrix from $H_{n',n'}$ we may assume $\vartheta = 1$ and $\alpha = 0 = \delta$ and if we additionally multiply with a suitable matrix from $H_{n,m}$ we can work with $\rho = 0 = \iota$. We choose the matrix $g := \begin{pmatrix} 0 & \beta & \gamma \\ 0 & \varepsilon & \theta \\ 1 & 0 & 0 \end{pmatrix}$ with $deg(\gamma) = n + n'$ is the maximal possible one and $det(g) = f$ as a representative for the double coset of M. Then, it follows

$$g^{-1}H_{n',n'}g =$$
$$\left\{ \begin{pmatrix} 0 & 0 & 1 \\ \frac{\theta}{f} & -\frac{\gamma}{f} & 0 \\ -\frac{\varepsilon}{f} & \frac{\beta}{f} & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & a_7 \end{pmatrix} \begin{pmatrix} 0 & \beta & \gamma \\ 0 & \varepsilon & \theta \\ 1 & 0 & 0 \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & a_7 \end{pmatrix} \in H_{n',n'} \right\} =$$
$$\left\{ \begin{pmatrix} a_7 & 0 & 0 \\ \frac{a_3\theta - a_6\gamma}{f} & \frac{\theta(a_1\beta + a_2\varepsilon) - \gamma(a_4\beta + a_5\varepsilon)}{f} & \frac{\theta(a_1\gamma + a_2\theta) - \gamma(a_4\gamma + a_5\theta)}{f} \\ \frac{a_6\beta - a_3\varepsilon}{f} & \frac{-\varepsilon(a_1\beta + a_2\varepsilon) + \beta(a_4\beta + a_5\varepsilon)}{f} & \frac{-\varepsilon(a_1\gamma + a_2\theta) + \beta(a_4\gamma + a_5\theta)}{f} \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & a_7 \end{pmatrix} \in H_{n',n'} \right\}.$$

We consider matrices in $g^{-1}H_{n',n'}g \cap H_{n,m}$: Here the equalities $a_3\varepsilon = a_6\beta$ and $a_3\theta = a_6\gamma$ hold. Since θ and γ are non-zero coprime polynomials it follows that γ divides a_3. With $deg(\gamma) = n + n' > n' \geq deg(a_3)$ we conclude $a_3 = 0$ and hence $a_6 = 0$. Moreover, the equality $a_2\varepsilon^2 = (a_5 - a_1)\beta\varepsilon + a_4\beta^2$ implies $a_2 = 0 = a_4$ and $a_5 = a_1$, because β and ε are coprime polynomials that are not both in the field k. We obtain

$g^{-1}H_{n',n'}g \cap H_{n,m} = \left\{ \begin{pmatrix} a_7 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_1 \end{pmatrix} \mid a_1, a_7 \in k^\times \right\}$. Thus $|g^{-1}H_{n',n'}g \cap H_{n,m}| = q - 1$.

According to Lemma 3.5.12 2.a) we get $|\Upsilon^{\vartheta\neq 0}_{n',n',n,m}| = (q-1)q^{n'+m'+2}(q-1)(q^2-1)q^{d+2n-m+2} = (q-1)^2(q^2-1)q^{2d+4}$. This leads to

$$\frac{|\Upsilon^{\vartheta\neq 0}_{n',n',n,m}|}{|H_{n',n'}||H_{n,m}|}|g^{-1}H_{n',n'}g \cap H_{n,m}| = \frac{(q-1)^2(q^2-1)q^{2d+4}}{(q^2-1)(q^2-q)q^{2n'+2}(q-1)^2q^{2n+3}}(q-1) = q^{d-m-2}.$$

Hence, in the case where the lower left entry of a representative of the double coset is a non-zero element in k, we get q^{d-m-2} double cosets. Then we have to sum up the number of double cosets in both cases (cf. 1(r)i and 2(r)i).

d) $n' > m' > 0$: The degree restraints for the entries of M are given by

$$deg(\alpha) \leq n',\ deg(\beta) \leq n+n'-m,\ deg(\gamma) \leq n+n',\ deg(\delta) \leq m',\ deg(\varepsilon) \leq n-m+m',\ deg(\theta) \leq n+m',\ deg(\vartheta) \leq 0,\ deg(\rho) \leq n-m,\ deg(\iota) \leq n.$$

In this case we see again, that there are two different types of double cosets, i.e. double cosets with representative, that has a zero entry in the lower left corner or double cosets, where the representatives have non-zero entries in the lower left corner.

Suppose $\vartheta = 0$. Moreover, we may assume $\lambda = 1$, i.e. $det(M) = \iota(\alpha\varepsilon - \delta\beta) - \rho(\alpha\theta - \delta\gamma) = f$. Then we compute

$$M^{-1}H_{n',m'}M =$$
$$\left\{ \begin{pmatrix} \frac{\varepsilon\iota-\rho\theta}{f} & \frac{\gamma\rho-\beta\iota}{f} & \frac{\beta\theta-\varepsilon\gamma}{f} \\ -\frac{\delta\iota}{f} & \frac{\alpha\iota}{f} & \frac{\delta\gamma-\alpha\theta}{f} \\ \frac{\delta\rho}{f} & -\frac{\alpha\rho}{f} & \frac{\alpha\varepsilon-\delta\beta}{f} \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{pmatrix} \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ 0 & \rho & \iota \end{pmatrix} \;\middle|\; \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{pmatrix} \in H_{n',m'} \right\}$$

$$= \left\{ \begin{pmatrix} \frac{(\varepsilon\iota-\rho\theta)(a_1\alpha+a_2\delta)+(\gamma\rho-\beta\iota)(a_4\delta)}{f} & \frac{(\varepsilon\iota-\rho\theta)(a_1\beta+a_2\varepsilon+a_3\rho)+(\gamma\rho-\beta\iota)(a_4\varepsilon+a_5\rho)+(\beta\theta-\varepsilon\gamma)a_6\rho}{f} & \frac{(\varepsilon\iota-\rho\theta)(a_1\gamma+a_2\theta+a_3\iota)+(\gamma\rho-\beta\iota)(a_4\theta+a_5\iota)+(\beta\theta-\varepsilon\gamma)a_6\iota}{f} \\ \frac{-\delta\iota(a_1\alpha+a_2\delta)+\alpha\iota(a_4\delta)}{f} & \frac{-\delta\iota(a_1\beta+a_2\varepsilon+a_3\rho)+\alpha\iota(a_4\varepsilon+a_5\rho)+(\delta\gamma-\alpha\theta)a_6\rho}{f} & \frac{-\delta\iota(a_1\gamma+a_2\theta+a_3\iota)+\alpha\iota(a_4\theta+a_5\iota)+(\delta\gamma-\alpha\theta)a_6\iota}{f} \\ \frac{\delta\rho(a_1\alpha+a_2\delta)-\alpha\rho(a_4\delta)}{f} & \frac{\delta\rho(a_1\beta+a_2\varepsilon+a_3\rho)-\alpha\rho(a_4\varepsilon+a_5\rho)+(\alpha\varepsilon-\delta\beta)a_6\rho}{f} & \frac{\delta\rho(a_1\gamma+a_2\theta+a_3\iota)-\alpha\rho(a_4\theta+a_5\iota)+(\alpha\varepsilon-\delta\beta)a_6\iota}{f} \end{pmatrix} \;\middle|\; \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{pmatrix} \in H_{n',m'} \right\}.$$

Now we intersect this set with $H_{n,m}$: The lower left entry has to be zero, i.e. $(a_1 - a_4)\alpha + a_2\delta = 0$. Since α and δ are coprime polynomials and at least one of these two polynomials has its maximal possible degree, we deduce $a_1 = a_4$ and $a_2 = 0$. Moreover, we have $\delta\rho(a_1\beta+a_2\varepsilon+a_3\rho)-\alpha\rho(a_4\varepsilon+a_5\rho)+(\alpha\varepsilon-\delta\beta)a_6\rho = 0$, which implies $0 = (a_6 - a_1)(\alpha\varepsilon - \delta\beta) + \rho(a_3\delta - a_5\alpha)$. Because of the Irreducibility of f, we conclude that $\alpha\varepsilon - \delta\beta$ and ρ are coprime polynomials and both are non-zero. Together with the condition that at least one of these two polynomials has its maximal possible degree, we obtain $a_6 = a_1$ and $a_3\delta = a_5\alpha$. Since α and δ are coprime polynomials, where at least one of them has its maximal possible degree, there exists some element $b \in k$, such that $a_3 = b\alpha$ and $a_5 = b\delta$. This leads to

$$M^{-1}H_{n',m'}M \cap H_{n,m} = \left\{ \begin{pmatrix} a_1 & 0 & b\alpha \\ 0 & a_1 & b\delta \\ 0 & 0 & a_1 \end{pmatrix} \mid a_1 \in k^\times, b \in k \right\},$$

whence $|M^{-1}H_{n',m'}M \cap H_{n,m}| = q$. From Lemma 3.5.12 2.a) we deduce $|\Upsilon^{\vartheta=0}_{n',m',n,m}| = (q-1)(q^2-1)q^{n'+m'-1}(q-1)(q^2-1)q^{d+2n-m+2} = (q-1)^2(q^2-1)^2q^{2d+1}$. Hence

$$\frac{|\Upsilon^{\vartheta=0}_{n',m',n,m}|}{|H_{n',m'}||H_{n,m}|}|M^{-1}H_{n',m'}M \cap H_{n,m}| = \frac{(q-1)^2(q^2-1)^2q^{2d+1}}{(q-1)^2q^{2n'+3}(q-1)^2q^{2n+3}}q = (q+1)^2q^{d-n'+m'-m-4}.$$

For $d \geq n'-m'+m+4 \geq 6$ we have $(q+1)^2q^{d-n'+m'-m-4}$ double cosets with representatives, where the lower left entry is zero.

Suppose $\vartheta \neq 0$. From the degree restraints of the entries of M, we see that we can multiply with a suitable matrix from $H_{n',m'}$ to get $\vartheta = 1$ and $\alpha = 0 = \delta$. Then we can multiply with a suitable matrix from $H_{n,m}$ to obtain $\rho = 0 = \iota$. This is the reason why we may choose $g := \begin{pmatrix} 0 & \beta & \gamma \\ 0 & \varepsilon & \theta \\ 1 & 0 & 0 \end{pmatrix}$ with $det(g) = f$ as representative for the double coset of M. We compute

$$g^{-1}H_{n',m'}g =$$
$$\left\{ \begin{pmatrix} 0 & 0 & 1 \\ \frac{\theta}{f} & -\frac{\gamma}{f} & 0 \\ -\frac{\varepsilon}{f} & \frac{\beta}{f} & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{pmatrix} \begin{pmatrix} 0 & \beta & \gamma \\ 0 & \varepsilon & \theta \\ 1 & 0 & 0 \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{pmatrix} \in H_{n',m'} \right\} =$$
$$\left\{ \begin{pmatrix} a_6 & 0 & 0 \\ \frac{a_3\theta - a_5\gamma}{f} & \frac{\theta(a_1\beta+a_2\varepsilon)-\gamma(a_4\varepsilon)}{f} & \frac{\theta(a_1\gamma+a_2\theta)-\gamma(a_4\theta)}{f} \\ \frac{a_5\beta - a_3\varepsilon}{f} & \frac{-\varepsilon(a_1\beta+a_2\varepsilon)+\beta(a_4\varepsilon)}{f} & \frac{-\varepsilon(a_1\gamma+a_2\theta)+\beta(a_4\theta)}{f} \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{pmatrix} \in H_{n',m'} \right\}.$$

The intersection with $H_{n,m}$ yields $a_5\beta = a_3\varepsilon$. Since β and ε are coprime polynomials and for degree reasons, we conclude $a_5 = 0 = a_3$. Furthermore, we get $a_2\varepsilon = (a_4 - a_1)\beta$ and hence $a_2 = 0$ and $a_1 = a_4$ for the same reasons as above. Therefore, the intersection is given by

$$g^{-1}H_{n',m'}g \cap H_{n,m} = \left\{ \begin{pmatrix} a_6 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_1 \end{pmatrix} \mid a_1, a_6 \in k^\times \right\}.$$

Thus $|g^{-1}H_{n',m'}g \cap H_{n,m}| = q-1$. Using Lemma 3.5.12 2.a) we have $|\Upsilon^{\vartheta\neq 0}_{n',m',n,m}| = (q-1)q^{n'+m'+2}(q-1)(q^2-1)q^{d+2n-m+2} = (q-1)^2(q^2-1)q^{2d+4}$ and with 2.6.8 we conclude

$$\frac{|\Upsilon^{\vartheta\neq 0}_{n',m',n,m}|}{|H_{n',m'}||H_{n,m}|}|g^{-1}H_{n',m'}g\cap H_{n,m}| = \frac{(q-1)^2(q^2-1)q^{2d+4}}{(q-1)^2q^{2n'+3}(q-1)^2q^{2n+3}}(q-1) = (q+1)q^{d-n'+m'-m-2},$$

which means there are $(q+1)q^{d-n'+m'-m-2}$ double cosets with representatives having a non-zero element in the lower left corner. Note that we have $d = n+n-m+n'+m' \geq 2+1+2+1 \geq 6$. Now sum up the number of double cosets in both cases (cf. 1(t)i and 2(t)i).

2. $n = m > 0$:

   a) $n' = m' = 0$: Here the degree $d = n$, $deg(\gamma) \leq n$, $deg(\theta) \leq n$ and $\alpha,\ \delta,\ \vartheta,\ \beta,\ \varepsilon$ and ρ are elements in the field k. Multiplication with a suitable matrix from $H_{0,0}$ yields $\alpha = 1 = \varepsilon$, $\delta = \vartheta = 0 = \beta = \rho$ and $\iota = f$. Hence
   $$\begin{pmatrix}1&0&\gamma\\0&1&\theta\\0&0&f\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&f\end{pmatrix}\begin{pmatrix}1&0&\gamma\\0&1&\theta\\0&0&1\end{pmatrix} \in H_{0,0}\begin{pmatrix}1&0&0\\0&1&0\\0&0&f\end{pmatrix}H_{n,n}.$$
   Therefore we obtain only one double coset in this case (cf. 1(a)ii and 2(a)ii).

   b) $n' > m' = 0$: In this case $d = n+n'$. Note that we necessarily have $d \geq 2$, since n and n' are both greater than zero. Moreover, the degree restraints for the entries of M imply $\delta,\vartheta,\varepsilon,\rho \in k$ and $deg(\alpha) \leq n'$, $deg(\beta) \leq n'$, $deg(\gamma) \leq d$, $deg(\theta) \leq n$, $deg(\iota) \leq n$. We may assume $\vartheta = 0$ and $\alpha = 0 = \beta$ (if necessary multiply with a suitable matrix from $H_{n',0}$). Moreover, we assume $det(M) = \gamma\delta\rho = -f$ (we choose $\lambda = -1$) and for degree reasons we can assume $\gamma = f$. We compute
   $$\begin{pmatrix}0&0&f\\\delta&\varepsilon&\theta\\0&\rho&\iota\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&0&f\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}0&\rho&\iota\\\delta&\varepsilon&\theta\\0&0&1\end{pmatrix} \in H_{n',0}\begin{pmatrix}0&0&f\\0&1&0\\1&0&0\end{pmatrix}H_{n,n}.$$
   Hence, we derive one double coset for this case (cf. 1(a)iv and 2(a)iv).

   c) $n' = m' > 0$: Then the degree of f is given by $d = n+2n'$. Furthermore, ϑ and ρ are elements in the field k. If we multiply with a suitable matrix from $H_{n,n}$ we may assume $\vartheta = 0 = \iota$ and $\rho = 1$. Multiplication with a suitable matrix from $H_{n',n'}$ implies $\varepsilon = 0 = \beta$. Hence, we choose $g := \begin{pmatrix}\alpha&0&\gamma\\\delta&0&\theta\\0&1&0\end{pmatrix}$ with $det(g) = f$ as representative for the double coset of M.

   We calculate
   $$g^{-1}H_{n',n'}g =$$
   $$\left\{\begin{pmatrix}-\frac{\theta}{f}&\frac{\gamma}{f}&0\\0&0&1\\\frac{\delta}{f}&-\frac{\alpha}{f}&0\end{pmatrix}\begin{pmatrix}a_1&a_2&a_3\\a_4&a_5&a_6\\0&0&a_7\end{pmatrix}\begin{pmatrix}\alpha&0&\gamma\\\delta&0&\theta\\0&1&0\end{pmatrix} \mid \begin{pmatrix}a_1&a_2&a_3\\a_4&a_5&a_6\\0&0&a_7\end{pmatrix} \in H_{n',n'}\right\} =$$

$$\left\{\begin{pmatrix} \frac{-\theta(a_1\alpha+a_2\delta)+\gamma(a_4\alpha+a_5\delta)}{f} & \frac{-a_3\theta+a_6\gamma}{f} & \frac{\gamma(a_4\gamma+a_5\theta)-\theta(a_1\gamma+a_2\theta)}{f} \\ 0 & a_7 & 0 \\ \frac{\delta(a_1\alpha+a_2\delta)-\alpha(a_4\alpha+a_5\delta)}{f} & \frac{a_3\delta-a_6\alpha}{f} & \frac{-\alpha(a_4\gamma+a_5\theta)+\delta(a_1\gamma+a_2\theta)}{f} \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & a_7 \end{pmatrix} \in H_{n',n'}\right\}.$$

Take the intersection with $H_{n,n}$: We have $a_3\delta = a_6\alpha$. Since α and δ are coprime polynomials, the degree restraints for α, δ, a_3 and a_6 imply that there exists some element $b \in k$, such that $a_3 = b\alpha$ and $a_6 = b\delta$. Moreover, we have the equality $(a_1 - a_5)\alpha\delta + a_2\delta^2 = a_4\alpha^2$ in the intersection. Since α and δ are coprime polynomials which are not both in the field k, we deduce $a_1 = a_5$ and $a_2 = 0 = a_4$. This leads to

$$g^{-1}H_{n',n'}g \cap H_{n,n} = \left\{\begin{pmatrix} a_1 & b & 0 \\ 0 & a_7 & 0 \\ 0 & 0 & a_1 \end{pmatrix} \mid a_1, a_7 \in k^\times \text{ and } b \in k\right\}.$$

We conclude $|g^{-1}H_{n',n'}g \cap H_{n,n}| = (q-1)q$. Because of $\kappa = 0$ and $0 = n - m < n$ we get by Lemma 3.5.13 2.b) that $|\Upsilon_{n',n',n,n}| = (q-1)(q^2-1)^2 q^{d+n+m'+n'+3} = (q-1)(q^2-1)^2 q^{2d+3}$. This implies

$$\frac{|\Upsilon_{n',n',n,n}|}{|H_{n',n'}||H_{n,n}|} \cdot |g^{-1}H_{n',n'}g \cap H_{n,n}| =$$
$$\frac{(q-1)(q^2-1)^2 q^{2d+3}}{(q^2-1)(q^2-q)q^{2n'+2}(q^2-1)(q^2-q)q^{2n+2}} \cdot (q-1)q = q^{d-n-2}$$

double cosets, if $d \geq n+2 \geq 3$ (cf. 1(f)i and 2(f)i).

d) $n' > m' > 0$: To solve this case we use the symmetry 3.7.4 and the solution for the case $n > m > 0$ and $n' > m' = 0$. We find $(q+1)q^{d-n-n'+m'-2}$ double cosets if $d = n' + m' + n \geq 4$ (cf. 1(m)i and 2(m)i).

3. $n > m = 0$:

a) $n' = m' = 0$: These Indices imply $d = 2n$ and

$$deg(\alpha) \leq 0,\ deg(\beta) \leq n,\ deg(\gamma) \leq n,\ deg(\delta) \leq 0,\ deg(\varepsilon) \leq n,\ deg(\theta) \leq n,\ deg(\vartheta) \leq 0,\ deg(\rho) \leq n,\ deg(\iota) \leq n.$$

When we multiply with suitable matrices from $H_{0,0}$ and $H_{n,0}$ we can choose $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & \theta \\ 0 & \rho & \iota \end{pmatrix}$ with $det(g) = f$ as representative for the double coset of M. We calculate

$$g^{-1}H_{0,0}g =$$
$$\left\{\begin{pmatrix} a_1 & a_2\varepsilon + a_3\rho & a_2\theta + a_3\iota \\ \frac{a_4\iota - a_7\theta}{f} & \frac{\iota(a_5\varepsilon+a_6\rho)-\theta(a_8\varepsilon+a_9\rho)}{f} & \frac{\iota(a_5\theta+a_6\iota)-\theta(a_8\theta+a_9\iota)}{f} \\ \frac{a_7\varepsilon - a_4\rho}{f} & \frac{-\rho(a_5\varepsilon+a_6\rho)+\varepsilon(a_8\varepsilon+a_9\rho)}{f} & \frac{-\rho(a_5\theta+a_6\iota)+\varepsilon(a_8\theta+a_9\iota)}{f} \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} \in H_{0,0}\right\}.$$

Intersect with $H_{n,0}$: We have $a_4\iota = a_7\theta$. Since ι and θ are coprime polynomials, which are both non-zero, and $a_4, a_7 \in k$ this implies $a_4 = 0 = a_7$. Because of $\begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_5 & a_6 \\ 0 & a_8 & a_9 \end{pmatrix} \in H_{0,0}$ we conclude $\begin{pmatrix} a_5 & a_6 \\ a_8 & a_9 \end{pmatrix} \in \mathrm{PGL}_2(k)$. Now define $h = \begin{pmatrix} \varepsilon & \theta \\ \rho & \iota \end{pmatrix}$. From our computation above we see that the lower right 2×2 block of a matrix in $g^{-1}H_{0,0}g \cap H_{n,0}$ is a matrix in $h^{-1}\,\mathrm{PGL}_2(k)h \cap \mathrm{PGL}_2(k)$. According to Remark 3.7.3 we have $|h\,\mathrm{PGL}_2(k)h^{-1} \cap \mathrm{PGL}_2(k)| = q+1$ if $d = 2$ and for $d \geq 4$ it is $|h\,\mathrm{PGL}_2(k)h^{-1} \cap \mathrm{PGL}_2(k)| = \begin{cases} 1 & \text{for } \frac{q^{d-2}-1}{q+1} \text{ double cosets} \\ q+1 & \text{for 1 double coset} \end{cases}$.
For the first row of a matrix in $g^{-1}H_{0,0}g \cap H_{n,0}$ there is $a_1 \in k^\times$ and $a_2, a_3 \in k$. Therefore, there are $(q-1)q^2$ possibilities for the first row. We conclude the following:

- for $d = 2$: $|g^{-1}H_{0,0}g \cap H_{n,0}| = (q+1) \cdot (q-1)q^2 = (q^2-1)q^2$
- for $d \geq 4$: $|g^{-1}H_{0,0}g \cap H_{n,0}| = \begin{cases} (q-1)q^2 & \text{for } \frac{q^{d-2}-1}{q+1} \text{ double cosets} \\ (q^2-1)q^2 & \text{for 1 double coset} \end{cases}$.

In this case $\kappa = 0$ and $n' = m' = 0$, hence Lemma 3.5.12 2.c) yields $|\Upsilon_{0,0,n,0}| = (q^3-1)(q-1)(q^2-1)q^{2d+2}$. By 2.6.8 we get $\frac{|\Upsilon_{0,0,n,0}|}{|H_{0,0}||H_{n,0}|} = \frac{(q^3-1)(q-1)(q^2-1)q^{2d+2}}{(q^3-1)(q^3-q)q^2(q^2-1)(q^2-q)q^{2n+2}} = \frac{q^{d-4}}{(q^2-1)}$.
We conclude:

- for $d = 2$:
  $\frac{|\Upsilon_{0,0,n,0}|}{|H_{0,0}||H_{n,0}|} \cdot |g^{-1}H_{0,0}g \cap H_{n,0}| = \frac{q^{-2}}{(q^2-1)} \cdot (q^2-1)q^2 = 1$. There is only one double coset in this case (cf. 1(x)ii).
- for $d \geq 4$:
  $\frac{|\Upsilon_{0,0,n,0}|}{|H_{0,0}||H_{n,0}|} = \frac{q^{d-4}}{(q^2-1)} = \frac{q^{d-2}-1}{q+1} \cdot \frac{1}{(q-1)q^2} + \frac{1}{(q^2-1)q^2}$. Whence we have $\frac{q^{d-2}-1}{q+1} + 1 = \frac{q(q^{d-3}+1)}{q+1}$ double cosets (cf. 1(y)ii and 2(x)ii).

b) $n' > m' = 0$: Due to the symmetry 3.7.4 we deduce from the solution of the case $n = m > 0$ and $n' = m' > 0$ that we have $q^{d-n'-2}$ double cosets for $d = n' + 2n \geq 3$ (cf. 1(d)ii and 2(d)ii).

c) $n' = m' > 0$: It is $d = 2n + 2n' \geq 4$ and $deg(\alpha) \leq n'$, $deg(\beta) \leq n + n'$, $deg(\gamma) \leq n + n'$, $deg(\delta) \leq n'$, $deg(\varepsilon) \leq n + n'$, $deg(\theta) \leq n + n'$, $deg(\vartheta) \leq 0$, $deg(\rho) \leq n$, $deg(\iota) \leq n$. Consider the multiplication with matrices in $H_{n',n'}$ from left and matrices in $H_{n,0}$ from right. Via this multiplication the lower left entry ϑ is only multiplied with some elements in $k^\times$. Therefore, we have exactly two different cases for a representative of a double coset: $\vartheta = 0$ or $\vartheta \neq 0$.

First case $\vartheta = 0$:

We start by choosing a suitable representative for the double coset. Again by multiplication with matrices from $H_{n',n'}$ from the left and matrices from $H_{n,0}$ from the right we can find a representative $g := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ 0 & \rho & \iota \end{pmatrix}$ for the double coset, such that only the entries on the diagonal of g have their respective maximal possible degree. With this representative we want to calculate the intersection of

$$H_{n',n'}g = \left\{ \begin{pmatrix} a_1\alpha + a_2\delta & a_1\beta + a_2\varepsilon + a_3\rho & a_1\gamma + a_2\theta + a_3\iota \\ a_4\alpha + a_5\delta & a_4\beta + a_5\varepsilon + a_6\rho & a_4\gamma + a_5\theta + a_6\iota \\ 0 & a_7\rho & a_7\iota \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & a_7 \end{pmatrix} \in H_{n',n'} \right\}$$

and

$$gH_{n,0} = \left\{ \begin{pmatrix} b_1\alpha & b_2\alpha + b_4\beta + b_6\gamma & b_3\alpha + b_5\beta + b_7\gamma \\ b_1\delta & b_2\delta + b_4\varepsilon + b_6\theta & b_3\delta + b_5\varepsilon + b_7\theta \\ 0 & b_4\rho + b_6\iota & b_5\rho + b_7\iota \end{pmatrix} \mid \begin{pmatrix} b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 \\ 0 & b_6 & b_7 \end{pmatrix} \in H_{n,0} \right\}.$$

In the intersection we have $(a_1 - b_1)\alpha + a_2\delta = 0 = a_4\alpha + (a_5 - b_1)\delta$. Now α and δ are coprime polynomials, hence α divides a_2 and $a_5 - b_1$. But a_2 and $a_5 - b_1$ are elements in the field and the polynomial α is of its maximal possible degree, in particular, $deg(\alpha) = n' > 0$. Therefore we obtain $a_2 = 0$ and $a_5 = b_1$. Since $\alpha \neq 0$ the two equations above give us $a_1 = b_1$ and $a_4 = 0$. The equations $(b_4 - a_7)\rho + b_6\iota = 0 = b_5\rho + (b_7 - a_7)\iota$ have to be fulfilled in the intersection $H_{n',n'}g \cap gH_{n,0}$. We know that ρ and ι are coprime polynomials and ι has degree $n > 0$. Whence, we find similar to the arguments above that $b_4 = a_7 = b_7$ and $b_6 = 0 = b_5$. Next we use the equation $a_4\beta + a_5\varepsilon + a_6\rho = b_2\delta + b_4\varepsilon + b_6\theta$, i.e. $b_2\delta + (b_4 - b_1)\varepsilon = a_6\rho$. Here we see that $deg(\rho) < n$, $deg(\delta) < n'$ and $deg(\varepsilon) = n + n'$ imply $b_1 = b_4$. Now there are the equations $a_3\rho = b_2\alpha$, $b_2\delta = a_6\rho$, $a_3\iota = b_3\alpha$ and $b_3\delta = a_6\iota$ left. Since ρ and ι are coprime polynomials α has to divide a_3. For degree reasons there has to exist an element $b \in k$ with $a_3 = b\alpha$. It follows $b_2 = b\rho$, $a_6 = b\delta$ and $b_3 = b\iota$. We derive

$$H_{n',n'}g \cap gH_{n,0} = \left\{ g \begin{pmatrix} a & b\rho & b\iota \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \mid a \in k^\times \text{ and } b \in k \right\}.$$

Thus $|H_{n',n'}g \cap gH_{n,0}| = q$ and whence $|g^{-1}H_{n',n'}g \cap H_{n,0}| = q$. According to

Lemma 3.5.12 2.a) we have $|\Upsilon^{\vartheta=0}_{n',n',n,0}| = (q-1)(q^2-1)q^{n'+m'-1}(q-1)(q^2-1)q^{d+2n-m+2} = (q-1)^2(q^2-1)^2q^{2d+1}$. Therefore,

$$\frac{|\Upsilon^{\vartheta=0}_{n',n',n,0}|}{|H_{n',n'}||H_{n,0}|} \cdot |g^{-1}H_{n',n'}g \cap H_{n,0}| = \frac{(q-1)^2(q^2-1)^2q^{2d+1}}{(q^2-1)(q^2-q)q^{2n'+2}(q^2-1)(q^2-q)q^{2n+2}} \cdot q = q^{d-4}.$$

So there are q^{d-4} double cosets, where the lower left entry of a matrix in this double coset is equal to zero.

Second case $\vartheta \neq 0$: If we multiply with a suitable matrix from $H_{n',n'}$ we may assume $\vartheta = 1$ and $\alpha = 0 = \delta$. Furthermore, multiplication with a suitable matrix from $H_{n,0}$ yields $\rho = 0 = \iota$. Hence, we choose the matrix $g := \begin{pmatrix} 0 & \beta & \gamma \\ 0 & \varepsilon & \theta \\ 1 & 0 & 0 \end{pmatrix}$ with $deg(\gamma) = n + n'$ is of its maximal possible degree and $det(g) = f$ as a representative for the double coset of M. It follows

$$g^{-1}H_{n',n'}g =$$
$$\left\{ \begin{pmatrix} 0 & 0 & 1 \\ \frac{\theta}{f} & -\frac{\gamma}{f} & 0 \\ -\frac{\varepsilon}{f} & \frac{\beta}{f} & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & a_7 \end{pmatrix} \begin{pmatrix} 0 & \beta & \gamma \\ 0 & \varepsilon & \theta \\ 1 & 0 & 0 \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & a_7 \end{pmatrix} \in H_{n',n'} \right\} =$$
$$\left\{ \begin{pmatrix} a_7 & 0 & 0 \\ \frac{a_3\theta - a_6\gamma}{f} & \frac{\theta(a_1\beta+a_2\varepsilon)-\gamma(a_4\beta+a_5\varepsilon)}{f} & \frac{\theta(a_1\gamma+a_2\theta)-\gamma(a_4\gamma+a_5\theta)}{f} \\ \frac{a_6\beta - a_3\varepsilon}{f} & \frac{-\varepsilon(a_1\beta+a_2\varepsilon)+\beta(a_4\beta+a_5\varepsilon)}{f} & \frac{-\varepsilon(a_1\gamma+a_2\theta)+\beta(a_4\gamma+a_5\theta)}{f} \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & a_7 \end{pmatrix} \in H_{n',n'} \right\}.$$

The next step is to consider the intersection $g^{-1}H_{n',n'}g \cap H_{n,0}$: Here the equalities $a_3\theta = a_6\gamma$ and $a_3\varepsilon = a_6\beta$ hold. Since θ and γ are coprime polynomials it follows that γ has to divide a_3, but then $deg(\gamma) = n + n' > n \geq deg(a_3)$ implies $a_3 = 0$.Whence it follows $0 = a_6$. Moreover, the matrix $\begin{pmatrix} a_1 & a_2 \\ a_4 & a_5 \end{pmatrix}$ is an element in the group $\mathrm{PGL}_2(k)$ and if we define $h := \begin{pmatrix} \beta & \gamma \\ \varepsilon & \theta \end{pmatrix}$, the lower right 2×2 block in the intersection $g^{-1}H_{n',n'}g \cap H_{n,0}$ is nothing else than $h^{-1}\mathrm{PGL}_2(k)h \cap \mathrm{PGL}_2(k)$. By 3.7.3 we deduce for $d \geq 4$: $|h^{-1}\mathrm{PGL}_2(k)h \cap \mathrm{PGL}_2(k)| = \begin{cases} 1 & \text{for } \frac{q^{d-2}-1}{q+1} \text{ double cosets} \\ q+1 & \text{for 1 double coset} \end{cases}$. For the first column in the intersection we have $a_7 \in k^\times$, hence there are $q-1$ possibilities for this column. In total we have $|g^{-1}H_{n'n'}g \cap H_{n,0}| = \begin{cases} q-1 & \text{for } \frac{q^{d-2}-1}{q+1} \text{ double cosets} \\ q^2-1 & \text{for 1 double coset} \end{cases}$.

Using Lemma 3.5.12 2.a) we get $|\Upsilon^{\vartheta\neq 0}_{n',n',n,0}| = (q-1)q^{n'+m'+2}(q-1)(q^2-1)q^{d+2n-m+2} = (q-1)^2(q^2-1)q^{2d+4}$. We compute

$$\frac{|\Upsilon^{\vartheta\neq 0}_{n',n',n,0}|}{|H_{n',n'}||H_{n,0}|} = \frac{(q-1)^2(q^2-1)q^{2d+4}}{(q^2-1)(q^2-q)q^{2n'+2}(q^2-1)(q^2-q)q^{2n+2}} = \frac{q^{d-2}}{(q^2-1)} = \frac{q^{d-2}-1}{q+1}\frac{1}{q-1} + 1\frac{1}{(q^2-1)}.$$

This implies $\frac{q^{d-2}-1}{q+1}+1 = \frac{q(q^{d-3}+1)}{q+1}$ double cosets in the case, where the lower left entry of the matrices in this double coset is a non-zero element in the field k. Sum up the total number of double cosets in both cases (cf. 1(z)i and 2(y)i).

d) $n' > m' > 0$: Now we can use again the symmetry 3.7.4, this time together with the solutions of case $n > m > 0$ and $n' = m' > 0$. This leads to $d = 2n+n'+m' \geq 5$ and we make the case distinction for the elements in the double coset, wether they have a zero entry in the lower left corner or not, i.e. $\vartheta = 0$ or $\vartheta \neq 0$. For $\vartheta = 0$ we obtain $(q+1)q^{d-n'+m'-4}$ double cosets and the number of double cosets with $\vartheta \neq 0$ is given by $q^{d-n'+m'-2}$. The sum of these two numbers gives us the total number of double cosets in this case (cf. 1(s)i and 2(s)i).

4. $n = m = 0$:

a) $n' = m' = 0$: This is not possible, since $d > 0$.

b) $n' > m' = 0$: By symmetry 3.7.4 and case $n = m > 0$ and $n' = m' = 0$ we obtain that for $d = n'$ we have one double coset (cf. 1(a)i and 2(a)i).

c) $n' = m' > 0$: We can use again the symmetry 3.7.4, together with the solutions of case $n > m = 0$ and $n' = m' = 0$. Then we find $d = 2n'$ and there is only one double coset for $d = 2$ (cf. 1(x)i) and if $d \geq 4$ we get $\frac{q(q^{d-3}+1)}{q+1}$ double cosets (cf. 1(y)i and 2(x)i).

d) $n' > m' > 0$: According to the symmetry 3.7.4 and the solution of case $n > m > 0$ and $n' = m' = 0$ we know that $d = n'+m'$ and we find $q^{d-n'+m'-2}$ double cosets in this case, if the degree satisfies $d \geq n'-m'+2 \geq 3$ (cf. 1(d)i and 2(d)i).

3.7.3. The case $\kappa > 0$, i.e. $d > 2n+n'-m+m'$

In order to solve this case we distinguish again between 16 subcases, corresponding to all possible combinations for the Indices n, m and n', m'.

1. $n > m > 0$:

a) $n' > m' > 0$: In this case we have the following degree restraints for the entries of M:

$$deg(\alpha) \leq \kappa + n',\ deg(\beta) \leq \kappa + n - m + n',\ deg(\gamma) \leq \kappa + n + n',\ deg(\delta) \leq \kappa + m',\ deg(\varepsilon) \leq \kappa + n - m + m',\ deg(\theta) \leq \kappa + n + m',\ deg(\vartheta) \leq \kappa,\ deg(\rho) \leq \kappa + n - m,\ deg(\iota) \leq \kappa + n.$$

By multiplication with elements from $H_{n,m}$ and $H_{n',m'}$ we can not change the degree of ϑ, since this multiplication induces for ϑ a multiplication with some non-zero element in k. Moreover, the multiplication with elements from $H_{n,m}$ and $H_{n',m'}$ induces for the entries δ and ρ just a multiplication by some non-zero element in k and adding a scalar multiple of ϑ. Hence we consider the following six subcases:

- If ϑ, δ and ρ have all not their respective maximal possible degree, then we choose as representative the matrix $g := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix}$ with α, ε and ι of their respective maximal possible degree and all other entries of g have less degree than it is possible. Furthermore, if $\vartheta \neq 0$ we choose g such that $a\delta = b\vartheta$ with $a \in k$ and $b \in k[t]$, $deg(b) \leq m'$ implies $a = 0$ or $\delta = 0$. Similar for ρ instead of δ.
- If ϑ and ρ have not their respective maximal possible degree, but δ has its maximal possible degree, we choose $g := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix}$, where δ, β and ι are of their respective maximal possible degree, $deg(\alpha) < deg(\delta)$ and the two entries θ and γ have less than their respective maximal possible degree. Moreover, if we have $\vartheta \neq 0$ we choose g such that $a\rho = b\vartheta$ with $a \in k$ and $b \in k[t]$, $deg(b) \leq n - m$ implies $a = 0$ or $\rho = 0$. (If the degree of α is greater than the degree of δ we can multiply with the matrix $\begin{pmatrix} 1 & -\alpha_{deg(\alpha)} t^{deg(\alpha)-\kappa-m'} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in H_{n',m'}$ from the left to decrease the degree of α by 1. When we do this multiplication successive $deg(\alpha) - \kappa - m' + 1$ times, where the degree of α in the matrix is in each step one less than the degree in the step before, then we can assume $deg(\alpha) < deg(\delta)$). Note that we do not say anything about the degree of ε. This is necessary to make our above assumption that $a\rho = b\vartheta$ implies $a = 0$ or $\rho = 0$.
- If ϑ and δ have not their respective maximal possible degree, but ρ has its maximal possible degree, we choose $g := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix}$, where ρ, θ and α are of their respective maximal possible degree, $deg(\iota) < deg(\rho)$ and

the two entries β and γ have less than their respective maximal possible degree. Moreover, if we have $\vartheta \neq 0$ we choose g such that $a\delta = b\vartheta$ with $a \in k$ and $b \in k[t]$, $deg(b) \leq m'$ implies $a = 0$ or $\delta = 0$. (Analogously to the previous case we can multiply successive with a suitable matrix in $H_{n,m}$ from the right to find $deg(\iota) < deg(\rho)$).

- If ϑ has not its maximal possible degree and δ and ρ are both of their respective maximal possible degree, we take as representative the matrix $g := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix}$, where δ, ρ and γ are the only entries of their respective maximal possible degree.
- If ϑ and ε have their respective maximal possible degree, we choose $g := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix}$ with ϑ, ε and γ of their respective maximal possible degree and all other entries have less degree than it is possible as representative for the double coset. Additionally we choose $\delta = 0$ or it is not possible to find some non-zero element $a \in k$ with $a\delta = b\vartheta$ for some suitable $b \in k[t]$. Similar, we choose $\rho = 0$ or $a\rho = b\vartheta$ with $a \in k, b \in k[t]$ implies $a = 0$.
- If ϑ has its maximal possible degree and ε has not its maximal possible degree, we take $g := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix}$ with ϑ, β and θ of their respective maximal possible degree and all other entries of less degree compared to the maximal possible degree, respectively, as representative for the double coset. As in the previous case we choose additionally $\delta = 0$ or it is not possible to find some non-zero element $a \in k$ with $a\delta = b\vartheta$ for some suitable $b \in k[t]$. And analogously, we choose $\rho = 0$ or $a\rho = b\vartheta$ with $a \in k, b \in k[t]$ implies $a = 0$.

Next we intersect the sets

$$H_{n',m'}g = \left\{ \begin{pmatrix} a_1\alpha + a_2\delta + a_3\vartheta & a_1\beta + a_2\varepsilon + a_3\rho & a_1\gamma + a_2\theta + a_3\iota \\ a_4\delta + a_5\vartheta & a_4\varepsilon + a_5\rho & a_4\theta + a_5\iota \\ a_6\vartheta & a_6\rho & a_6\iota \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{pmatrix} \in H_{n',m'} \right\}$$

and

$$gH_{n,m} = \left\{ \begin{pmatrix} b_1\alpha & b_2\alpha + b_4\beta & b_3\alpha + b_5\beta + b_6\gamma \\ b_1\delta & b_2\delta + b_4\varepsilon & b_3\delta + b_5\varepsilon + b_6\theta \\ b_1\vartheta & b_2\vartheta + b_4\rho & b_3\vartheta + b_5\rho + b_6\iota \end{pmatrix} \mid \begin{pmatrix} b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 \\ 0 & 0 & b_6 \end{pmatrix} \in H_{n,m} \right\}.$$

- Consider the first case for the representative g, i.e. the only entries of g with their respective maximal possible degree are those on the diagonal. Then the equation $a_1\alpha + a_2\delta + a_3\vartheta = b_1\alpha$ implies $a_1 = b_1$ and hence $a_2\delta + a_3\vartheta = 0$. From $a_4\varepsilon + a_5\rho = b_2\delta + b_4\varepsilon$ we deduce $a_4 = b_4$ and whence $a_5\rho = b_2\delta$. Using $a_6\iota = b_3\vartheta + b_5\rho + b_6\iota$ we get $a_6 = b_6$ and $0 = b_3\vartheta + b_5\rho$.

  If $\vartheta = 0$ it follows from the determinant of g and the Irreducibility of f that $\delta \neq 0 \neq \rho$. With $a_4\delta + a_5\vartheta = b_1\delta$ we have $a_4 = b_1$ and $a_6\rho = b_2\vartheta + b_4\rho$ yields $a_6 = b_4$. Furthermore, $0 = b_3\vartheta + b_5\rho$ implies $a_5 = 0$ and $a_2\delta + a_3\vartheta = 0$ yields $a_2 = 0$. The remaining equations are $a_5\rho = b_2\delta$, $a_3\rho = b_2\alpha$, $a_5\iota = b_3\delta$ and $a_3\iota = b_3\alpha$. Since ι and ρ are coprime polynomials we conclude that α divides a_3, but $deg(\alpha) = \kappa + n' > n' \geq deg(a_3)$, which implies $a_3 = 0$ and hence $a_5 = 0 = b_2 = b_3$.

  If $\vartheta \neq 0$, we consider the four possibilities for δ and ρ: Suppose $\delta = 0 = \rho$. From $a_6\vartheta = b_1\vartheta$ we get $a_6 = b_1$. Using $a_4\delta + a_5\vartheta = b_1\delta$ we deduce $a_5 = 0$. Moreover, $a_2\delta + a_3\vartheta = 0$ yields $a_3 = 0$, $a_6\rho = b_2\vartheta + b_4\rho$ implies $b_2 = 0$ and $0 = b_3\vartheta + b_5\rho$ gives $b_3 = 0$. Then the equation $(a_1 - b_4)\beta + a_2\varepsilon = 0$ is left. Since ε and β are coprime polynomials and $deg(\varepsilon) = \kappa + n - m + m' > 0$ we deduce $a_2 = 0$ and $a_1 = b_4$. From $b_5\varepsilon = 0$ we get $b_5 = 0$.

  For $\delta \neq 0 = \rho$ we have $a_6 = b_1$ because of $a_6\vartheta = b_1\vartheta$. From the equation $(a_4 - b_1)\delta + a_5\vartheta = 0$ we conclude $a_5 = 0$ and $a_4 = b_1$ according to our choice of the representative g. This implies $b_2\vartheta = 0$ and whence $b_2 = 0$. Some remaining equations are $a_2\delta + a_3\vartheta = 0 = a_2\varepsilon + a_3\rho$ and $b_3\vartheta + b_5\rho = 0 = b_3\delta + b_5\varepsilon$. Due to Lemma 3.5.7 a) we obtain $a_2 = 0 = a_3$ and $b_3 = 0 = b_5$.

  Suppose $\delta = 0 \neq \rho$. We have again $a_6\vartheta = b_1\vartheta$, i.e. $a_6 = b_1$. The equation $a_4\delta + a_5\vartheta = b_1\delta$ implies $a_5 = 0$. Using $a_6\rho = b_2\vartheta + b_4\rho$ and our choice of g we conclude $a_6 = b_4$ and $b_2 = 0$. Similar to the previous case, by Lemma 3.5.7 a), the remaining equations $a_2\delta + a_3\vartheta = 0 = a_2\varepsilon + a_3\rho$ and $b_3\vartheta + b_5\rho = 0 = b_3\delta + b_5\varepsilon$ yield $a_2 = 0 = a_3$ and $b_3 = 0 = b_5$.

  For $\delta \neq 0 \neq \rho$ the equations $a_6\vartheta = b_1\vartheta$, $a_4\delta + a_5\vartheta = b_1\delta$ and $a_6\rho = b_2\vartheta + b_4\rho$ imply $a_6 = b_1 = a_4$, $a_5 = 0$ and $b_2 = 0$. Similar to the both cases above the remaining equations are $a_2\delta + a_3\vartheta = 0 = a_2\varepsilon + a_3\rho$ and $b_3\vartheta + b_5\rho = 0 = b_3\delta + b_5\varepsilon$, which yields with Lemma 3.5.7 a) that $a_2 = 0 = a_3$ and $b_3 = 0 = b_5$.

- The second case is $deg(\vartheta)$ and $deg(\rho)$ are not maximal, respectively, but the degree of δ is maximal. From $a_4\delta + a_5\vartheta = b_1\delta$ we deduce $a_4 = b_1$ and $a_5\vartheta = 0$. Furthermore $a_1\alpha + a_2\delta + a_3\vartheta = b_1\alpha$ implies $a_2 = 0$ and $(b_1 - a_1)\alpha = a_3\vartheta$ due to our choice of the representative g. The equation $a_6\iota = b_3\vartheta + b_5\rho + b_6\iota$ yields $a_6 = b_6$ and $b_3\vartheta + b_5\rho = 0$.

If $\vartheta = 0$, then we necessarily have $\rho \neq 0 \neq \alpha$, because the polynomial f is irreducible. Therefore we conclude from the equations $(a_6 - b_4)\rho = 0 = b_5\rho$ and $(a_1 - b_1)\alpha = 0$ that $a_6 = b_4$, $b_5 = 0$ and $a_1 = b_1$. Hence we have the equation $(a_4 - b_4)\beta + a_3\rho = b_2\alpha$, which is only possible for $a_4 = b_4$ and $a_3\rho = b_2\alpha$. The remaining equations are $a_3\rho = b_2\alpha$, $a_5\rho = b_2\delta$, $a_5\iota = b_3\delta$ and $a_3\iota = b_3\alpha$. Since α and δ are coprime polynomials we derive from these equations that ι has to divide b_3. Now $deg(\iota) = \kappa + n > n \geq deg(b_3)$ implies $b_3 = 0$ and whence $a_5 = 0 = a_3 = b_2$.

If $\vartheta \neq 0$ we can use $(a_6 - b_1)\vartheta = 0 = a_5\vartheta$ to obtain $a_6 = b_1$ and $a_5 = 0$.

For $\rho \neq 0 \neq \vartheta$ the equation $(a_6 - b_4)\rho = b_2\vartheta$ gives us $a_6 = b_4$ and $b_2 = 0$, since we have chosen g such that this equation is only possible for $a_6 = b_4$ and $b_2 = 0$. From the equation $(a_1 - b_4)\beta + a_3\rho = 0$ we can deduce $a_1 = b_4$ and $a_3\rho = 0$, i.e. $a_3 = 0$. We have now the equations $b_3\vartheta + b_5\rho = 0 = b_3\delta + b_5\varepsilon$. When we apply Lemma 3.5.7 a) to these two equations we find $b_3 = 0 = b_5$.

For $\rho = 0 \neq \vartheta$ we have $b_2\vartheta = 0 = b_3\vartheta$ and hence $b_2 = 0 = b_3$. Moreover, since f is irreducible we conclude from the determinant of g that ε has to be non-zero. Therefore the equations $(a_4 - b_4)\varepsilon = 0 = b_5\varepsilon$ imply $a_4 = b_4$ and $b_5 = 0$. Now there is the equation $(a_1 - b_4)\beta = 0$ left, which means $a_1 = b_4$. Then $a_3 = 0$ because of $a_3\vartheta = 0$.

- Now we compute the intersection for a representative of the third case, i.e. ϑ and δ have not their respective maximal possible degree, but ρ has. $a_1\alpha + a_2\delta + a_3\vartheta = b_1\alpha$ implies $a_1 = b_1$ and $a_2\delta + a_3\vartheta = 0$. We use $a_6\rho = b_2\vartheta + b_4\rho$ to observe that $a_6 = b_4$ and $0 = b_2\vartheta$. From $a_6\iota = b_3\vartheta + b_5\rho + b_6\iota$ we deduce $b_5 = 0$ and $(a_6 - b_6)\iota = b_3\vartheta$. The equation $a_4\theta + a_5\iota = b_3\delta + b_5\varepsilon + b_6\theta$ yields $a_4 = b_6$ and $a_5\iota = b_3\delta$.

  If $\vartheta = 0$, then $\delta \neq 0 \neq \iota$, because the polynomial f is irreducible. This is the reason why $(a_4 - b_1)\delta = 0 = a_2\delta$ and $(a_6 - b_6)\iota = 0$ imply $a_4 = b_1$, $a_2 = 0$ and $a_6 = b_6$. Hence there are the equations $a_5\iota = b_3\delta$, $a_3\rho = b_2\alpha$, $a_5\rho = b_2\delta$ and $a_3\iota = b_3\alpha$ left. Since the polynomials α and δ are coprime we obtain that ρ divides b_2. With $deg(\rho) = \kappa + n - m > n - m \geq deg(b_2)$ we arrive at $b_2 = 0$ and whence $a_3 = 0 = b_3 = a_5$.

  Suppose $\vartheta \neq 0$. Due to the equations $(a_6 - b_1)\vartheta = 0 = b_2\vartheta$ we find $a_6 = b_1$ and $b_2 = 0$.

  For $\vartheta \neq 0 = \delta$ we have $a_3\vartheta = 0 = a_5\vartheta$ which means $a_3 = 0 = a_5$. Since f is irreducible we know from the determinant of g that ε has to be non-zero. Therefore we can use $(a_4 - b_4)\varepsilon = 0 = a_2\varepsilon$ to conclude $a_4 = b_4$ and $a_2 = 0$. Now $b_3\vartheta = 0$ gives us $b_3 = 0$.

For $\vartheta \neq 0 \neq \delta$ we can use our choice of the representative g together with the equation $a_5\vartheta = (b_1 - a_4)\delta$ to derive $b_1 = a_4$ and $a_5 = 0$. Next we apply Lemma 3.5.7 a) to the equations $a_2\delta + a_3\vartheta = 0 = a_2\varepsilon + a_3\rho$ to get $a_2 = 0 = a_3$. The remaining equation $b_3\vartheta = 0$ is equivalent to $b_3 = 0$.

- Next we want to compute the intersection for the fourth case, which means ϑ has not its maximal possible degree and ρ and δ are both polynomials of their respective maximal possible degree. The equation $a_4\delta + a_5\vartheta = b_1\delta$ gives $a_4 = b_1$ and $a_5\vartheta = 0$ and from $a_6\rho = b_2\vartheta + b_4\rho$ we deduce $a_6 = b_4$ and $0 = b_2\vartheta$. Furthermore $a_1\gamma + a_2\theta + a_3\iota = b_3\alpha + b_5\beta + b_6\gamma$ implies $a_1 = b_6$.

  Suppose $\vartheta = 0$, then α and δ as well as ρ and ι are coprime polynomials. Moreover $deg(\rho)$ and $deg(\delta)$ are both greater than zero. Thus $(a_1 - b_1)\alpha + a_2\delta = 0$ and $(a_6 - b_6)\iota = b_5\rho$ yield $a_1 = b_1$, $a_2 = 0$, $a_6 = b_6$ and $b_5 = 0$. Therefore the remaining equations are $a_5\rho = b_2\delta$, $a_3\rho = b_2\alpha$, $a_5\iota = b_3\delta$ and $a_3\iota = b_3\alpha$. Hence δ divides a_5 and $deg(\delta) = \kappa + m' > m' \geq deg(a_5)$, which implies $a_5 = 0$ and whence $b_3 = 0 = a_3 = b_2$.

  If $\vartheta \neq 0$ we conclude from $(a_6 - b_1)\vartheta = 0$ that $a_6 = b_1$ and the above equations $a_5\vartheta = 0 = b_2\vartheta$ imply $a_5 = 0 = b_2$. Consider the equations $(a_1 - b_1)\alpha + a_2\delta + a_3\vartheta = 0 = (a_1 - b_1)\beta + a_2\varepsilon + a_3\rho$.

  Due to Lemma 3.5.7 b) we have $a_1 = b_1$ and $a_2 = 0 = a_3$. Then we apply again Lemma 3.5.7 a) to the remaining equations $b_3\delta + b_5\varepsilon = 0 = b_3\alpha + b_5\beta$ to get $b_3 = 0 = b_5$.

- The fifth case is ϑ and ε have their respective maximal possible degree. We conclude from $(a_6 - b_1)\vartheta = 0$ that $a_6 = b_1$ and the equation $a_4\varepsilon + a_5\rho = b_2\delta + b_4\varepsilon$ implies $a_4 = b_4$ and $a_5\rho = b_2\delta$. According to $a_1\gamma + a_2\theta + a_3\iota = b_3\alpha + b_5\beta + b_6\gamma$ we have $a_1 = b_6$.

  Suppose $\delta = 0 = \rho$, then α and ϑ as well as ι and ϑ are coprime polynomials and the degree of ϑ is greater than zero. Therefore the equations $(a_1 - b_1)\alpha + a_3\vartheta = 0$ and $(a_6 - b_6)\iota = b_3\vartheta$ yield $a_1 = b_1$, $a_3 = 0$, $a_6 = b_6$ and $b_3 = 0$. Moreover, $a_5\vartheta = 0 = b_2\vartheta$ implies $a_5 = 0 = b_2$. Since $\varepsilon \neq 0$ we derive from $a_2\varepsilon = 0$ that $a_2 = 0$. To finish this case we use $b_5\varepsilon = 0$ to get $b_5 = 0$.

  For $\delta \neq 0 = \rho$ we use our certain choice of g to derive from $(a_4 - b_1)\delta + a_5\vartheta = 0$ and $b_2\vartheta = 0$ that $a_4 = b_1$ and $a_5 = 0 = b_2$. Since $\vartheta \in k[t] \setminus k$ and ι are coprime polynomials we deduce from $(a_6 - b_6)\iota = b_3\vartheta$ that $a_6 = b_6$ and $b_3 = 0$. Hence there are the equations $a_2\delta + a_3\vartheta = 0 = a_2\varepsilon + a_3\rho$ left. Due to Lemma 3.5.7 a) this means $a_2 = 0 = a_3$. Now $b_5\varepsilon = 0$ yields $b_5 = 0$.

If $\delta = 0 \neq \rho$ the equations $a_5\vartheta = 0$ and $(a_6 - b_4)\rho = b_2\vartheta$ imply $a_5 = 0 = b_2$ and $a_6 = b_4$ due to our choice of g. Moreover, α and ϑ are coprime polynomials and $deg(\vartheta) > 0$, which yield $a_1 = b_1$ and $a_3 = 0$ from the equation $(a_1 - b_1)\alpha + a_3\vartheta = 0$. Using $a_2\varepsilon = 0$ and $\varepsilon \neq 0$ we obtain $a_2 = 0$. With Lemma 3.5.7 a) the two remaining equations $b_3\delta + b_5\varepsilon = 0 = b_3\vartheta + b_5\rho$ yield $b_3 = 0 = b_5$.

Suppose $\delta \neq 0 \neq \rho$, then with our certain choice of g we deduce from the equations $(a_4 - b_1)\delta + a_5\vartheta = 0$ and $(a_6 - b_4)\rho = b_2\vartheta$ that $a_4 = b_1$, $a_5 = 0 = b_2$ and $a_6 = b_4$. We have then the equations $(a_1 - b_1)\alpha + a_2\delta + a_3\vartheta = 0 = (a_1 - b_1)\beta + a_2\varepsilon + a_3\rho$. Using Lemma 3.5.7 b) we conclude $a_1 = b_1$ and $a_2 = 0 = a_3$. We have again the equations $b_3\delta + b_5\varepsilon = 0 = b_3\alpha + b_5\beta$ left, which means by Lemma 3.5.7 a) that $b_3 = 0 = b_5$.

- The last case is when ϑ has its maximal possible degree, but ε has not its maximal possible degree. From $(a_6 - b_1)\vartheta = 0$ we deduce $a_6 = b_1$. According to $a_1\beta + a_2\varepsilon + a_3\rho = b_2\alpha + b_4\beta$ we conclude $a_1 = b_4$ and $a_2\varepsilon + a_3\rho = b_2\alpha$. As $a_4\theta + a_5\iota = b_3\delta + b_5\varepsilon + b_6\theta$ we get $a_4 = b_6$ and $a_5\iota = b_3\delta + b_5\varepsilon$.

  Suppose $\delta = 0 = \rho$. This implies that α and ϑ as well as ι and ϑ are coprime polynomials. Together with $deg(\vartheta) > 0$ and $(a_1 - b_1)\alpha + a_3\vartheta = 0$ and $(a_6 - b_6)\iota = b_3\vartheta$ we conclude $a_1 = b_1$, $a_3 = 0 = b_3$ and $a_6 = b_6$. Furthermore we have $a_5\vartheta = 0 = b_2\vartheta$, which gives us $a_5 = 0 = b_2$. Since f is irreducible we know that ε is non-zero in this case, hence we deduce from $a_2\varepsilon = 0 = b_5\varepsilon$ that $a_2 = 0 = b_5$.

  For $\delta \neq 0 = \rho$ the choice of g and the equations $(a_4 - b_1)\delta + a_3\vartheta = 0$ and $b_2\vartheta = 0$ imply $a_5 = 0 = b_2$ and $a_4 = b_1$. Since $\varepsilon \neq 0$ we deduce $a_4 = b_4$ from $(a_4 - b_4)\varepsilon = 0$. Using Lemma 3.5.7 a) for the equations $b_3\vartheta + b_5\rho = 0 = b_3\delta + b_5\varepsilon$ we get $b_3 = 0 = b_5$. Now we can again apply Lemma 3.5.7 a) to $a_2\varepsilon + a_3\rho = 0 = a_2\delta + a_3\vartheta$ to find $a_2 = 0 = a_3$.

  If $\delta = 0 \neq \rho$ we derive from $a_5\vartheta = 0$ and $(a_6 - b_4)\rho = b_2\vartheta$ with our choice of g that $a_5 = 0 = b_2$ and $a_6 = b_4$. Furthermore we know that α and ϑ are coprime polynomials, which implies together with $deg(\vartheta) > 0$ and $(a_1 - b_1)\alpha + a_3\vartheta = 0$ that $a_3 = 0$ and $a_1 = b_1$. Now we deduce from the fact that f is irreducible, the entry ε is non-zero and that is why we get $a_4 = b_4$ and $a_2 = 0$ from $(a_4 - b_4)\varepsilon = 0$ and $a_2\varepsilon = 0$. Using again Lemma 3.5.7 a) we obtain $b_3 = 0 = b_5$ from $b_3\alpha + b_5\beta = 0 = b_3\vartheta + b_5\rho$.

  The last possibility is $\delta \neq 0 \neq \rho$. Then by our certain choice of g the equations $(a_6 - b_4)\rho = b_2\vartheta$ and $(a_4 - b_1)\delta + a_3\vartheta = 0$ lead to $a_6 = b_4$, $b_2 = 0 = a_3$ and $a_4 = b_1$. The remaining equations are $a_2\delta + a_3\vartheta = 0 = a_2\varepsilon + a_3\rho$ and $b_3\delta + b_5\varepsilon = 0 = b_3\vartheta + b_5\rho$. Applying Lemma 3.5.7 a) to these equations we get $a_2 = 0 = a_3$ and $b_3 = 0 = b_5$.

In all these cases we computed that the intersection $H_{n',m'}g \cap gH_{n,m}$ is equal to the set $\{a_1 g \mid a_1 \in k^\times\}$. Therefore the intersection in this case is trivial, i.e. $|H_{n',m'}g \cap gH_{n,m}| = \frac{q-1}{q-1} = 1$.

According to Lemma 3.5.12 3. we have $|\Upsilon_{n',m',n,m}| = (q-1)^2(q^2-1)^2(q^2+q+1)q^{2d}$. We conclude that there are

$$\frac{|\Upsilon_{n',m',n,m}|}{|H_{n',m'}||H_{n,m}|} = \frac{(q-1)^2(q^2-1)^2(q^2+q+1)q^{2d}}{(q-1)^2q^{2n'+3}(q-1)^2q^{2n+3}} = (q+1)^2(q^2+q+1)q^{2d-2n'-2n-6}$$

double cosets in this case, if $d \geq n + n' + 3 \geq 2 + 2 + 3 = 7$ (cf. 1(u)i and 2(u)i).

b) $n' = m' > 0$: In this case the degree restraints are given by

$$deg(\alpha) \leq \kappa + n',\ deg(\beta) \leq \kappa + n - m + n',\ deg(\gamma) \leq \kappa + n + n',\ deg(\delta) \leq \kappa + n',\ deg(\varepsilon) \leq \kappa + n - m + n',\ deg(\theta) \leq \kappa + n + n',\ deg(\vartheta) \leq \kappa,\ deg(\rho) \leq \kappa + n - m,\ deg(\iota) \leq \kappa + n.$$

If we consider the multiplication with matrices from $H_{n',n'}$ and $H_{n,m}$ for the double coset we see that we can multiply ϑ only with some non-zero elements from the field k, i.e. the degree of ϑ does not change by this multiplication. Moreover, ρ can be multiplied with some non-zero element from k and we may add some multiple of ϑ. This observation leads to the following case distinction:

- If ϑ and ρ have both not their respective maximal possible degree we choose as representative for the double coset the matrix $g := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix}$, where the only entries with their respective maximal possible degree are those on the diagonal of g. Moreover, if $\vartheta \neq 0$ we choose g in such a way that $\rho = 0$ or the equation $a\rho = b\vartheta$ for $a \in k, b \in k[t]$ with $deg(b) \leq n-m$ implies $a = 0$.
- If ϑ has not its maximal possible degree, but ρ is of its maximal possible degree we take the representative $g := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix}$ with ρ, α and θ of their respective maximal possible degree and all other entries of g have less degree than it is possible. Furthermore we choose g such that we have $\varepsilon = 0$ or the equation $a\varepsilon = b\rho$ for $a \in k, b \in k[t]$ with $deg(b) \leq n'$ yields $a = 0$.
- If ϑ is some polynomial of its maximal possible degree we can choose

$g := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix}$ with ϑ, ε and γ of their respective maximal possible degree and all other entries have not their respective maximal possible degree as representative for the double coset. Additionally, we assume that $\alpha = 0$ or we can not find some $a \in k^{\times}$, $b \in k[t]$, $deg(b) \leq n'$ with $a\alpha = b\vartheta$.

Now we want to compute the intersection of

$$H_{n',n'}g = \left\{ \begin{pmatrix} a_1\alpha + a_2\delta + a_3\vartheta & a_1\beta + a_2\varepsilon + a_3\rho & a_1\gamma + a_2\theta + a_3\iota \\ a_4\alpha + a_5\delta + a_6\vartheta & a_4\beta + a_5\varepsilon + a_6\rho & a_4\gamma + a_5\theta + a_6\iota \\ a_7\vartheta & a_7\rho & a_7\iota \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & a_7 \end{pmatrix} \in H_{n',n'} \right\}$$

with

$$gH_{n,m} = \left\{ \begin{pmatrix} b_1\alpha & b_2\alpha + b_4\beta & b_3\alpha + b_5\beta + b_6\gamma \\ b_1\delta & b_2\delta + b_4\varepsilon & b_3\delta + b_5\varepsilon + b_6\theta \\ b_1\vartheta & b_2\vartheta + b_4\rho & b_3\vartheta + b_5\rho + b_6\iota \end{pmatrix} \mid \begin{pmatrix} b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 \\ 0 & 0 & b_6 \end{pmatrix} \in H_{n,m} \right\}$$

for all possible representatives g.

- For $deg(\vartheta)$ and $deg(\rho)$ are both not maximal we use the equation $a_4\alpha + a_5\delta + a_6\vartheta = b_1\delta$ to deduce $a_4 = 0$ and $(a_5 - b_1)\delta + a_6\vartheta = 0$. From $a_1\alpha + a_2\delta + a_3\vartheta = b_1\alpha$ we get $a_1 = b_1$ and $a_2\delta + a_3\vartheta = 0$. According to $a_4\beta + a_5\varepsilon + a_6\rho = b_2\delta + b_4\varepsilon$ we see that $a_5 = b_4$ and $a_6\rho = b_2\delta$. The equation $a_7\iota = b_3\vartheta + b_5\rho + b_6\iota$ implies $a_7 = b_6$ and $0 = b_3\vartheta + b_5\rho$.

  If $\vartheta = 0$ we conclude from $(a_5 - b_1)\delta + a_6\vartheta = 0$, $a_2\delta + a_3\vartheta = 0$, $(a_7 - b_4)\rho = 0$ and $0 = b_3\vartheta + b_5\rho$ that we have $a_5 = b_1$, $a_2 = 0$, $a_7 = b_4$ and $b_5 = 0$, since $\delta \neq 0 \neq \rho$ according to the determinant of g and the Irreducibility of f. The remaining equations are given by $a_3\rho = b_2\alpha$, $a_6\rho = b_2\delta$, $a_6\iota = b_3\delta$ and $a_3\iota = b_3\alpha$. Since α and δ are coprime polynomials we deduce that ι has to divide b_3. With $deg(\iota) = \kappa + n > n \geq deg(b_3)$ it follows $b_3 = 0$ and hence $a_6 = 0 = b_2 = a_3$.

  Suppose $\vartheta \neq 0$ then $(a_7 - b_1)\vartheta = 0$ implies $a_7 = b_1$.

  For $\rho = 0$ the equations $b_2\vartheta = 0 = b_3\vartheta$ yield $b_2 = 0 = b_3$. Since ε and β are coprime polynomials and $deg(\varepsilon) > 0$ we conclude from $(a_1 - b_4)\beta + a_2\varepsilon = 0$ that $a_1 = b_4$ and $a_2 = 0$. Using the equations $a_3\vartheta = 0 = a_6\vartheta$ we see that $a_3 = 0 = a_6$. Now $b_5\varepsilon = 0$ implies $b_5 = 0$.

  For $\rho \neq 0$ we deduce due to our certain choice of g from $(a_7 - b_4)\rho = b_2\vartheta$ that $a_7 = b_4$ and $b_2 = 0$. Then $a_6\rho = 0$, which means $a_6 = 0$. Thus the equations $b_3\delta + b_5\varepsilon = 0 = b_3\vartheta + b_5\rho$ and $a_2\theta + a_3\iota = 0 = a_2\delta + a_3\vartheta$ yield $b_3 = 0 = b_5$ and $a_2 = 0 = a_3$ if we apply Lemma 3.5.7 a).

- If ϑ has not its maximal possible degree and ρ is a polynomial of its maximal possible degree we conclude again $a_4 = 0$ and $(a_5-b_1)\delta+a_6\vartheta = 0$ from $a_4\alpha + a_5\delta + a_6\vartheta = b_1\delta$ as well as $a_1 = b_1$ and $a_2\delta + a_3\vartheta = 0$ from $a_1\alpha + a_2\delta + a_3\vartheta = b_1\alpha$. Furthermore the equation $(a_7 - b_4)\rho = b_2\vartheta$ yields $a_7 = b_4$ and $0 = b_2\vartheta$. Using $a_4\gamma + a_5\theta + a_6\iota = b_3\delta + b_5\varepsilon + b_6\theta$ we get $a_5 = b_6$ and $a_6\iota = b_3\delta + b_5\varepsilon$.

  Suppose $\vartheta = 0$, then by the Irreducibility of f we know that $\delta \neq 0$, which implies $b_1 = a_5$ and $a_2 = 0$ because of the equations $(b_1 - a_5)\delta = 0 = a_2\delta$. Due to the fact that ι and ρ are coprime polynomials and the degree of ρ is greater than zero we conclude from $(a_7 - b_6)\iota = b_5\rho$ that $a_7 = b_6$ and $b_5 = 0$. The remaining equations are $a_3\rho = b_2\alpha$, $a_6\rho = b_2\delta$, $a_6\iota = b_3\delta$ and $a_3\iota = b_3\alpha$. This implies α divides a_3, since ι and ρ are coprime. Now $deg(\alpha) = \kappa + n' > n' \geq deg(a_3)$ yields $a_3 = 0$ and whence $b_3 = 0 = a_6 = b_2$.

  Is $\vartheta \neq 0$ we know from $(a_7 - b_1)\vartheta = 0 = b_2\vartheta$ that $a_7 = b_1$ and $b_2 = 0$.

  For $\varepsilon = 0$ the polynomials ρ and β are coprime. This yields with $deg(\rho) > 0$ that $a_6 = 0 = a_3$ and $a_1 = b_4$ using the equations $a_6\rho = 0 = (a_1 - b_4)\beta + a_3\rho$. Thus we have $(b_1 - a_5)\delta = 0 = a_2\delta$, which means $b_1 = a_5$ and $a_2 = 0$, since $\delta \neq 0$ because otherwise f is reducible. By Lemma 3.5.7 a) we derive $b_3 = 0 = b_5$ from $b_3\alpha + b_5\beta = 0 = b_3\vartheta + b_5\rho$.

  If $\varepsilon \neq 0$ we can deduce from $(a_5 - b_4)\varepsilon + a_6\rho = 0$ that $a_5 = b_4$ and $a_6 = 0$ according to our choice of the representative g. Applying Lemma 3.5.7 a) to the remaining equations $b_3\alpha + b_5\beta = 0 = b_3\vartheta + b_5\rho$ and $a_2\varepsilon + a_3\rho = 0 = a_2\delta + a_3\vartheta$ gives $b_3 = 0 = b_5$ and $a_2 = 0 = a_3$.

- If ϑ is a polynomial of its maximal possible degree we have $a_7 = b_1$ because of $(a_7 - b_1)\vartheta = 0$. According to $a_4\beta + a_5\varepsilon + a_6\rho = b_2\delta + b_4\varepsilon$ we deduce $a_5 = b_4$ and $a_4\beta + a_6\rho = b_2\delta$. From $a_1\beta + a_2\varepsilon + a_3\rho = b_2\alpha + b_4\beta$ we know $a_2 = 0$. Furthermore $a_1\gamma + a_2\theta + a_3\iota = b_3\alpha + b_5\beta + b_6\gamma$ implies $a_1 = b_6$ and $a_3\iota = b_3\alpha + b_5\beta$.

  Suppose $\rho = 0$, then $b_2\vartheta = 0$ yields $b_2 = 0$. Since ϑ and ι are coprime polynomials and the degree of ϑ is greater zero, the equation $(a_7 - b_6)\iota = b_3\vartheta$ implies $a_7 = b_6$ and $b_3 = 0$. Thus we have $(a_1 - b_4)\beta = 0 = a_4\beta$, which gives $a_1 = b_4$ and $a_4 = 0$, because $\beta \neq 0$ follows from the determinant of g and the Irreducibility of f. Now we have $a_6\vartheta = 0 = a_3\vartheta$ and hence $a_6 = 0 = a_3$. Applying Lemma 3.5.7 a) to $b_3\vartheta + b_5\rho = 0 = b_3\alpha + b_5\beta$ implies $b_3 = 0 = b_5$.

  Suppose $\rho \neq 0$. For $\alpha = 0$ we know that ρ and ϑ are coprime polynomials. Together with $deg(\vartheta) > 0$ we deduce from $(a_7 - b_4)\rho = b_2\vartheta$ that $a_7 = b_4$ and $b_2 = 0$. Thus $a_6\vartheta = 0 = a_3\vartheta$, which means $a_6 = 0 = a_3$. This implies $a_4\beta = 0 = (a_1 - b_4)\beta$ and hence $a_4 = 0$ and $a_1 = b_4$, since we have $\beta \neq 0$ according to the determinant of g and the Irreducibility of f. Due to Lemma 3.5.7 a) we conclude from $b_3\alpha + b_5\beta = 0 = b_3\delta + b_5\varepsilon$ that $b_3 = 0 = b_5$.

For $\alpha \neq 0$ the equation $(a_1 - b_1)\alpha + a_3\vartheta = 0$ implies $a_1 = b_1$ and $a_3 = 0$. Using three times Lemma 3.5.7 a) we deduce from $b_2\alpha = (a_1 - b_4)\beta = (b_1 - b_4)\beta$ and $b_2\vartheta = (a_7 - b_4)\rho = (b_1 - b_4)\rho$ that $b_2 = 0$ and $b_1 = b_4$. Then there are the equations $a_4\alpha + a_6\vartheta = 0 = a_4\beta + a_6\rho$ who lead to $a_4 = 0 = a_6$. The remaining equations $b_3\vartheta + b_5\rho = 0 = b_3\alpha + b_5\beta$ yield $b_3 = 0 = b_5$.

So we obtain $H_{n',n'}g \cap gH_{n,m} = \{a_1 g \mid a_1 \in k^\times\}$ is trivial. Hence $|H_{n',n'}g \cap gH_{n,m}| = 1$. Due to Lemma 3.5.12 3. we know $|\Upsilon_{n',n',n,m}| = (q-1)^2(q^2-1)^2(q^2+q+1)q^{2d}$. Thus we compute

$$\frac{|\Upsilon_{n',n',n,m}|}{|H_{n',n'}||H_{n,m}|} = \frac{(q-1)^2(q^2-1)^2(q^2+q+1)q^{2d}}{(q^2-1)(q^2-q)q^{2n'+2}(q-1)^2q^{2n+3}} = (q+1)(q^2+q+1)q^{2d-2n'-2n-6}$$

as number of double cosets if $d \geq n' + n + 3 \geq 6$ (cf. 1(v)i and 2(v)i).

c) $n' > m' = 0$: The degree restraints for the entries of M are given by

$deg(\alpha) \leq \kappa + n'$, $deg(\beta) \leq \kappa + n - m + n'$, $deg(\gamma) \leq \kappa + n + n'$, $deg(\delta) \leq \kappa$, $deg(\varepsilon) \leq \kappa + n - m$, $deg(\theta) \leq \kappa + n$, $deg(\vartheta) \leq \kappa$, $deg(\rho) \leq \kappa + n - m$ and $deg(\iota) \leq \kappa + n$.

In this case we distinguish two cases for the representative of the double coset.

- If ϑ and δ have both not their respective maximal possible degree we choose as representative the matrix $g := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix}$, such that only the entries on the diagonal of g have their respective maximal possible degree. Furthermore we choose g such that if $\vartheta \neq 0$ we have $\delta = 0$ or δ is not a scalar multiple of ϑ.
- If ϑ or δ has its respective maximal possible degree and the degree of ε is its maximal possible one, we take the representative $g := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix}$ with ϑ, ε and γ of their respective maximal possible degree and all other entries have less than their respective possible degree.
- If ϑ or δ has its respective maximal possible degree and the degree of ε is not its maximal possible one, we take the representative $g := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix}$

with ϑ, β and θ of their respective maximal possible degree and all other entries have less than their respective possible degree.

We consider now the intersection between

$$H_{n',0}g = \left\{ \begin{pmatrix} a_1\alpha + a_2\delta + a_3\vartheta & a_1\beta + a_2\varepsilon + a_3\rho & a_1\gamma + a_2\theta + a_3\iota \\ a_4\delta + a_5\vartheta & a_4\varepsilon + a_5\rho & a_4\theta + a_5\iota \\ a_6\delta + a_7\vartheta & a_6\varepsilon + a_7\rho & a_6\theta + a_7\iota \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & a_6 & a_7 \end{pmatrix} \in H_{n',0} \right\}$$

and

$$gH_{n,m} = \left\{ \begin{pmatrix} b_1\alpha & b_2\alpha + b_4\beta & b_3\alpha + b_5\beta + b_6\gamma \\ b_1\delta & b_2\delta + b_4\varepsilon & b_3\delta + b_5\varepsilon + b_6\theta \\ b_1\vartheta & b_2\vartheta + b_4\rho & b_3\vartheta + b_5\rho + b_6\iota \end{pmatrix} \mid \begin{pmatrix} b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 \\ 0 & 0 & b_6 \end{pmatrix} \in H_{n,m} \right\}.$$

- Suppose ϑ and δ have not their respective maximal possible degree. Then the equation $a_1\alpha + a_2\delta + a_3\vartheta = b_1\alpha$ yields $a_1 = b_1$ and $a_2\delta + a_3\vartheta = 0$. From $a_6\varepsilon + a_7\rho = b_2\vartheta + b_4\rho$ we conclude $a_6 = 0$ and $(a_7 - b_4)\rho = b_2\vartheta$. Using $a_4\varepsilon + a_5\rho = b_2\delta + b_4\varepsilon$ we see that $a_4 = b_4$ and $a_5\rho = b_2\delta$. By $a_6\theta + a_7\iota = b_3\vartheta + b_5\rho + b_6\iota$ we have $a_7 = b_6$.

  For $\vartheta = 0$ it follows $\delta \neq 0 \neq \rho$ since f is irreducible. According to the equations $(a_4 - b_1)\delta = 0 = (a_7 - b_4)\rho$ and $a_2\delta = 0 = b_5\rho$ we deduce $a_4 = b_1$, $a_7 = b_4$, $a_2 = 0 = b_5$. The remaining equations are $a_5\rho = b_2\delta$, $a_3\rho = b_2\alpha$, $a_5\iota = b_3\delta$ and $a_3\iota = b_3\alpha$. Since α and δ are coprime polynomials we derive from these equations that ι divides b_3. Because of $deg(\iota) = \kappa + n > n \geq deg(b_3)$ we know $b_3 = 0$. Hence $a_5 = 0 = a_3 = b_2$.

  Is $\vartheta \neq 0 = \delta$ then we deduce $b_1 = a_7$, $a_5 = 0 = a_3$ from the equations $(a_7 - b_1)\vartheta = 0 = a_5\vartheta = a_3\vartheta$. The equation $(a_4 - b_6)\theta = b_5\varepsilon$ is left. Since ε and θ are coprime polynomials and $deg(\varepsilon) > 0$ we conclude $a_4 = b_6$ and $b_5 = 0$. This implies $b_3\vartheta = 0$, i.e. $b_3 = 0$. Now $a_2\varepsilon = 0$ implies $a_2 = 0$.

  For $\vartheta \neq 0 \neq \delta$ we have $b_1 = a_7$, since $(b_1 - a_7)\vartheta = 0$. Due to our choice of g we get from $(a_4 - b_1)\delta + a_5\vartheta = 0$ that $a_4 = b_1$ and $a_5 = 0$. This yields $b_2\delta = 0$ and hence $b_2 = 0$. Now we apply Lemma 3.5.7 a) to the equations $a_2\delta + a_3\vartheta = 0 = a_2\varepsilon + a_3\rho$ and $b_3\vartheta + b_5\rho = 0 = b_3\delta + b_5\varepsilon$ to derive $a_2 = 0 = a_3$ and $b_3 = 0 = b_5$.

- Suppose ϑ and ε have their respective maximal possible degree. From the equation $a_6\delta = (b_1 - a_7)\vartheta$ we deduce $a_7 = b_1$ and $a_6\delta = 0$. Using $(a_4 - b_1)\delta + a_5\vartheta = 0$ we get $a_5 = 0$ and $(a_4 - b_1)\delta = 0$. As in the previous case we use $a_4\varepsilon + a_5\rho = b_2\delta + b_4\varepsilon$ to see that $a_4 = b_4$ and $0 = b_2\delta$. According to $a_1\gamma + a_2\theta + a_3\iota = b_3\alpha + b_5\beta + b_6\gamma$ we have $a_1 = b_6$.

  If $\delta = 0$ we know that α and ϑ are coprime polynomials. Moreover the degree of ϑ is greater than zero. Therefore we deduce from $(a_1 - b_1)\alpha +$

$a_3\vartheta = 0$ that $a_1 = b_1$ and $a_3 = 0$. By similar arguments we conclude $a_4 = b_6$ and $b_5 = 0$ from $(a_4 - b_6)\theta = b_5\varepsilon$. The equations $a_6\varepsilon = b_2\vartheta$ and $a_2\varepsilon = b_2\alpha$ imply that ε divides b_2. For degree reasons this yields $b_2 = 0$ and whence $a_2 = 0 = a_6$. Now $b_3\vartheta = 0$ means $b_3 = 0$.

For $\delta \neq 0$ we have $a_4 = b_1$ and $b_2 = 0 = a_6$ since $(a_4 - b_1)\delta = 0 = b_2\delta = a_6\delta$. So we have the equations $(a_1 - b_1)\alpha + a_2\delta + a_3\vartheta = 0 = (a_1 - b_1)\beta + a_2\varepsilon + a_3\rho$. Next we apply Lemma 3.5.7 b) to obtain $a_1 = b_1$ and $a_2 = 0 = a_3$. Using again Lemma 3.5.7 a) we conclude from $b_3\vartheta + b_5\rho = 0 = b_3\delta + b_5\varepsilon$ that $b_3 = 0 = b_5$.

- We consider now the case where ϑ has its maximal possible degree, but ε is a polynomial with less degree than it is possible. By the equation $a_6\delta + a_7\vartheta = b_1\vartheta$ we conclude $a_7 = b_1$ and $a_6\delta = 0$. From $a_4\delta + a_5\vartheta = b_1\delta$ we derive $a_5 = 0$ and $(a_4 - b_1)\delta = 0$. Due to $a_1\beta + a_2\varepsilon + a_3\rho = b_2\alpha + b_4\beta$ we know that $a_1 = b_4$ and $a_2\varepsilon + a_3\rho = b_2\alpha$. Using the equation $a_4\theta + a_5\iota = b_3\delta + b_5\varepsilon + b_6\theta$ we obtain $a_4 = b_6$ and $b_3\delta + b_5\varepsilon = 0$

  For $\delta = 0$ we have necessarily $\varepsilon \neq 0$ since f is irreducible. Then the equations $b_5\varepsilon = 0 = (a_4 - b_4)\varepsilon$ lead to $b_5 = 0$ and $a_4 = b_4$. Due to the fact that $deg(\vartheta) > 0$ and the two polynomials ϑ and α are coprime we can use $(b_1 - a_1)\alpha = a_3\vartheta$ to find $a_1 = b_1$ and $a_3 = 0$. Then we consider the remaining equations $a_6\varepsilon = b_2\vartheta$, $a_2\theta = b_3\alpha$, $a_6\theta = b_3\vartheta$ and $a_2\varepsilon = b_2\alpha$. Since ε and θ are coprime polynomials we deduce that ϑ has to divide a_6. With $deg(\vartheta) = \kappa > 0 \geq deg(a_6)$ it follows that $a_6 = 0$ and hence $a_2 = b_2 = 0 = b_3$.

  For $\delta \neq 0$ we use $a_6\delta = 0 = (a_4 - b_1)\delta$ to obtain $a_6 = 0$ and $a_4 = b_1$. Now we apply Lemma 3.5.7 a) to the equations $b_2\vartheta = (a_7 - b_4)\rho = (b_1 - b_4)\rho$ and $b_2\delta = (a_4 - b_4)\varepsilon = (b_1 - b_4)\varepsilon$. Therefore we have $b_1 = b_4$ and $b_2 = 0$. Again by Lemma 3.5.7 a) we deduce from $b_3\vartheta + b_5\rho = 0 = b_3\delta + b_5\varepsilon$ and $a_2\delta + a_3\vartheta = 0 = a_2\varepsilon + a_3\rho$ that $b_3 = 0 = b_5$ and $a_2 = 0 = a_3$.

Now all cases for the representative of the double coset in this case yield that the intersection is given by $H_{n',0}g \cap gH_{n,m} = \{a_1 g \mid a_1 \in k^\times\}$. This implies $|H_{n',0}g \cap gH_{n,m}| = 1$ and with Lemma 3.5.12 3. we obtain

$$\frac{|\Upsilon_{n',0,n,m}|}{|H_{n',0}||H_{n,m}|} = \frac{(q-1)^2(q^2-1)^2(q^2+q+1)q^{2d}}{(q^2-1)(q^2-q)q^{2n'+2}(q-1)^2q^{2n+3}} = (q+1)(q^2+q+1)q^{2d-2n'-2n-6}$$

double cosets for $d \geq n' + n + 3 \geq 6$ (cf. 1(v)ii and 2(v)ii).

d) $n' = m' = 0$: Now we have the following degree restraints for the entries of the representative:

$$deg(\alpha) \leq \kappa,\ deg(\beta) \leq \kappa + n - m,\ deg(\gamma) \leq \kappa + n,\ deg(\delta) \leq \kappa,\ deg(\varepsilon) \leq \kappa + n - m,\ deg(\theta) \leq \kappa + n,\ deg(\vartheta) \leq \kappa,\ deg(\rho) \leq \kappa + n - m,\ deg(\iota) \leq \kappa + n.$$

By multiplication with suitable matrices from $H_{0,0}$ and $H_{n,m}$ we can take as representative for the double coset the matrix $g := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix}$, where the entries on the diagonal have their respective maximal possible degree and all other entries have less degree than it is possible. Furthermore we may assume if $\vartheta \neq 0$ then $\delta = 0$ or δ is not a scalar multiple of ϑ.

We consider the intersection of

$$H_{0,0}g = \left\{ \begin{pmatrix} a_1\alpha + a_2\delta + a_3\vartheta & a_1\beta + a_2\varepsilon + a_3\rho & a_1\gamma + a_2\theta + a_3\iota \\ a_4\alpha + a_5\delta + a_6\vartheta & a_4\beta + a_5\varepsilon + a_6\rho & a_4\gamma + a_5\theta + a_6\iota \\ a_7\alpha + a_8\delta + a_9\vartheta & a_7\beta + a_8\varepsilon + a_9\rho & a_7\gamma + a_8\theta + a_9\iota \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} \in H_{0,0} \right\}$$

and

$$gH_{n,m} = \left\{ \begin{pmatrix} b_1\alpha & b_2\alpha + b_4\beta & b_3\alpha + b_5\beta + b_6\gamma \\ b_1\delta & b_2\delta + b_4\varepsilon & b_3\delta + b_5\varepsilon + b_6\theta \\ b_1\vartheta & b_2\vartheta + b_4\rho & b_3\vartheta + b_5\rho + b_6\iota \end{pmatrix} \mid \begin{pmatrix} b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 \\ 0 & 0 & b_6 \end{pmatrix} \in H_{n,m} \right\}.$$

In this intersection we have the equation $a_7\alpha + a_8\delta + a_9\vartheta = b_1\vartheta$, which implies $a_7 = 0$ and $(b_1 - a_9)\vartheta = a_8\delta$. Moreover, $a_4\alpha + a_5\delta + a_6\vartheta = b_1\delta$ gives us $a_4 = 0$ and $a_6\vartheta = (b_1 - a_5)\delta$. From $a_1\alpha + a_2\delta + a_3\vartheta = b_1\alpha$ we deduce $a_1 = b_1$ and $a_2\delta + a_3\vartheta = 0$. Using $a_4\beta + a_5\varepsilon + a_6\rho = b_2\delta + b_4\varepsilon$ we conclude $a_5 = b_4$ and $a_6\rho = b_2\delta$. By the equation $a_7\beta + a_8\varepsilon + a_9\rho = b_2\vartheta + b_4\rho$ we know $a_8 = 0$ and $(a_9 - b_4)\rho = b_2\vartheta$. Due to $a_7\gamma + a_8\theta + a_9\iota = b_3\vartheta + b_5\rho + b_6\iota$ we have $a_9 = b_6$, $0 = b_3\vartheta + b_5\rho$.

If $\vartheta = 0$ we know that $\delta \neq 0 \neq \rho$ since f is irreducible, which yields together with $(b_1 - a_5)\delta = 0 = a_2\delta = (a_9 - b_4)\rho = b_5\rho$ that $b_1 = a_5$, $a_2 = 0 = b_5$ and $a_9 = b_4$. The remaining equations are $a_3\rho = b_2\alpha$, $a_3\iota = b_3\alpha$, $b_3\delta = a_6\iota$ and $a_6\rho = b_2\delta$. Since the polynomials α and δ are coprime we know that ι has to divide b_3. But then $deg(\iota) = \kappa + n > n \geq deg(b_3)$ implies $b_3 = 0$. This means $a_6 = 0 = b_2 = a_3$.

If $\vartheta \neq 0$ it follows from $(b_1 - a_9)\vartheta = 0$ that $a_9 = 0 = b_1$.

For $\delta = 0$ the equations $a_6\vartheta = 0 = a_3\vartheta$ imply $a_3 = 0 = a_6$. From $(a_5 - a_1)\theta = b_5\varepsilon$ we conclude $a_5 = a_1$ and $b_5 = 0$, because θ and $\varepsilon \in k[t] \setminus k$ are coprime polynomials. Then the equations $a_2\varepsilon = b_2\alpha$ and $a_2\theta = b_3\alpha$ yield that α has to divide a_2. But with $deg(\alpha) = \kappa > 0 \geq deg(a_2)$ this is only possible for $a_2 = 0$. Hence we have $b_2 = 0 = b_3$ due to the last two equations above.

For $\delta \neq 0$ our choice of the representative g allows us to deduce $a_6 = 0$ and $b_1 = a_5$ from the equation $a_6\vartheta = (b_1 - a_5)\delta$. Then we have $b_2\vartheta = 0$, i.e. $b_2 = 0$.

Next we apply Lemma 3.5.7 a) to the equations $a_2\delta + a_3\vartheta = 0 = a_2\theta + a_3\iota$ and $b_3\vartheta + b_5\rho = 0 = b_3\delta + b_5\varepsilon$ to obtain $a_2 = 0 = a_3$ and $b_3 = 0 = b_5$.

So the intersection is given by $H_{0,0}g \cap gH_{n,m} = \{a_1 g \mid a_1 \in k^\times\}$. Therefore $|H_{0,0}g \cap gH_{n,m}| = 1$ and by Lemma 3.5.12 3. we have

$$\frac{|\Upsilon_{0,0,n,m}|}{|H_{0,0}||H_{n,m}|} = \frac{(q-1)^2(q^2-1)^2(q^2+q+1)q^{2d}}{(q^3-1)(q^3-q)q^2(q-1)^2q^{2n+3}} = (q+1)q^{2d-2n-6}$$

double cosets if $d \geq n+3 \geq 5$ (cf. 1(n)i and 2(n)i).

2. $n = m > 0$:

   a) $n' > m' > 0$: This case follows by symmetry 3.7.4 from $n > m > 0$ and $n' > m' = 0$. Hence for a representative g we have $|H_{n',m'}g \cap gH_{n,n}| = 1$. Furthermore, we use Lemma 3.5.12 3. to compute

   $$\frac{|\Upsilon_{n',m',n,n}|}{|H_{n',m'}||H_{n,n}|} = \frac{(q-1)^2(q^2-1)^2(q^2+q+1)q^{2d}}{(q-1)^2q^{2n'+3}(q^2-1)(q^2-q)q^{2n+2}} = (q+1)(q^2+q+1)q^{2d-2n'-2n-6}$$

   double cosets in this case, if $d \geq n'+n+3 \geq 6$ (cf. 1(v)iii and 2(v)iii).

   b) $n' = m' > 0$: Now the degree restraints are the following:

   $$deg(\alpha) \leq \kappa + n',\ deg(\beta) \leq \kappa + n',\ deg(\gamma) \leq \kappa + n' + n,\ deg(\delta) \leq \kappa + n',\ deg(\varepsilon) \leq \kappa + n',\ deg(\theta) \leq \kappa + n + n',\ deg(\vartheta) \leq \kappa,\ deg(\rho) \leq \kappa,\ deg(\iota) \leq \kappa + n.$$

   If we multiply with matrices from $H_{n,n}$ or $H_{n',n'}$ we can only have linear combinations of ϑ and ρ for the two first entries in the last row of the matrix M. This leads to the following choices for the representatives of the double coset:

   - If ϑ or ρ has its respective maximal possible degree. Then we choose as representative for the double coset the matrix $g := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix}$, where the entries ϑ, ε and γ have their respective maximal possible degree and all other entries have not their respective maximal possible degree.
   - If ϑ and ρ are both not of their respective maximal possible degree we take the representative $g := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix}$, where the only entries with their respective maximal possible degree are those on the diagonal of g. Moreover, we can assume $deg(\vartheta) > deg(\rho)$.

   Now we want to compute the intersection of

$$H_{n',n'}g =$$
$$\left\{ \begin{pmatrix} a_1\alpha + a_2\delta + a_3\vartheta & a_1\beta + a_2\varepsilon + a_3\rho & a_1\gamma + a_2\theta + a_3\iota \\ a_4\alpha + a_5\delta + a_6\vartheta & a_4\beta + a_5\varepsilon + a_6\rho & a_4\gamma + a_5\theta + a_6\iota \\ a_7\vartheta & a_7\rho & a_7\iota \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & a_7 \end{pmatrix} \in H_{n',n'} \right\}$$

with

$$gH_{n,n} =$$
$$\left\{ \begin{pmatrix} b_1\alpha + b_4\beta & b_2\alpha + b_5\beta & b_3\alpha + b_6\beta + b_7\gamma \\ b_1\delta + b_4\varepsilon & b_2\delta + b_5\varepsilon & b_3\delta + b_6\varepsilon + b_7\theta \\ b_1\vartheta + b_4\rho & b_2\vartheta + b_5\rho & b_3\vartheta + b_6\rho + b_7\iota \end{pmatrix} \mid \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ 0 & 0 & b_7 \end{pmatrix} \in H_{n,n} \right\}$$

for the representatives g.

Here we have the equation $a_7\vartheta = b_1\vartheta + b_4\rho$. Since in both possible cases the degree of ϑ is greater than the degree of ρ we deduce from this equation that $a_7 = b_1$ and $b_4\rho = 0$. Similar argumentation yields from $a_7\rho = b_2\vartheta + b_5\rho$ that we have $b_2 = 0$ and $(a_7 - b_5)\rho = 0$.

- If the first case holds, i.e. we have $deg(\vartheta)$ is maximal, we conclude from $a_4\beta + a_5\varepsilon + a_6\rho = b_2\delta + b_5\varepsilon$ the equations $a_5 = b_5$ and $a_4\beta + a_6\rho = 0$. Using $a_1\beta + a_2\varepsilon + a_3\rho = b_2\alpha + b_5\beta$ we deduce $a_2 = 0$ and $(a_1 - b_5)\beta + a_3\rho = 0$. With $a_1\gamma + a_2\theta + a_3\iota = b_3\alpha + b_6\beta + b_7\gamma$ we know $a_1 = b_7$ and $a_3\iota = b_3\alpha + b_6\beta$.

  Suppose $\rho = 0$, then since f is irreducible we have $\beta \neq 0$ which implies $a_4 = 0$ from the equation $a_4\beta = 0$. Furthermore we get $a_1 = a_5$, because of the equation $(a_1 - a_5)\beta = 0$. Now we consider the equation $(a_7 - a_1)\iota = b_3\vartheta$. Since ϑ and ι are coprime polynomials and $deg(\vartheta) > 0 \geq deg(a_7 - a_1)$ we derive $a_7 = a_1$ and $b_3 = 0$. The remaining equations are $a_3\vartheta = b_4\beta$, $a_6\vartheta = b_4\varepsilon$, $a_6\iota = b_6\varepsilon$ and $a_3\iota = b_6\beta$. Since β and ε are coprime polynomials and $deg(\vartheta) > 0$ the first two equations are only possible for $b_4 = 0 = a_3 = a_6$, because otherwise ϑ is a common divisor of β and ε. Hence, we obtain by the remaining two equations that $b_6 = 0$.

  Suppose $\rho \neq 0$. This implies $b_4 = 0$ and $a_7 = b_5$, because of $b_4\rho = 0$ and $(a_7 - b_5)\rho = 0$. From the equations $(a_1 - a_5)\alpha + a_3\vartheta = 0 = (a_1 - a_5)\beta + a_3\rho$ and $a_4\alpha + a_6\vartheta = 0 = a_4\beta + a_6\rho$ we conclude with Lemma 3.5.7 a) that $a_1 = a_5$ and $a_3 = 0 = a_4 = a_6$. Now we have the remaining equations $b_3\vartheta + b_6\rho = 0 = b_3\delta + b_6\varepsilon$, which yield again by Lemma 3.5.7 a) that $b_3 = 0 = b_6$.

- If ϑ and ρ have not their respective maximal possible degree we derive from $a_1\alpha + a_2\delta + a_3\vartheta = b_1\alpha$ that $a_1 = b_1$ and $a_2\delta + a_3\vartheta = 0$. By $a_4\beta + a_5\varepsilon + a_6\rho = b_2\delta + b_5\varepsilon$ we get $a_5 = b_5$ and $a_4\beta + a_6\rho = 0$. Using

$a_7\iota = b_3\vartheta + b_6\rho + b_7\iota$ we have $a_7 = b_7$ and $0 = b_3\vartheta + b_6\rho$. With $a_1\beta + a_2\varepsilon + a_3\rho = b_2\alpha + b_5\beta$ we obtain $a_2 = 0$ and $(a_1 - b_5)\beta + a_3\rho = 0$.

Suppose $\rho = 0$. In this case β is non-zero and the equations $a_4\beta = 0 = (a_1 - b_5)\beta$ imply $a_4 = 0$ and $a_1 = b_5$. Because of $deg(\varepsilon) > deg(\vartheta) \geq 0$ (we have $deg(\vartheta) \geq 0$ since we assumed $deg(\vartheta) > deg(\rho)$) the equation $a_6\vartheta = b_4\varepsilon$ is only possible for $a_6 = 0 = b_4$. By $a_3\vartheta = 0$ we see that $a_3 = 0$. The remaining equations are $b_3\delta + b_6\varepsilon = 0 = b_3\alpha + b_6\beta = b_3\vartheta + b_6\rho$, which implies $b_3 = 0 = b_6$, since the columns of g are linearly independent.

Suppose $\rho \neq 0$. This yields $b_4 = 0$ and $a_7 = b_5$, since we have the equations $b_4\rho = 0 = (a_7 - b_5)\rho$. We obtain $a_3 = 0$ from $a_3\vartheta = 0$, since $deg(\vartheta) > deg(\rho)$ implies $\vartheta \neq 0$. Apply Lemma 3.5.7 a) to $a_4\alpha + a_6\vartheta = 0 = a_4\beta + a_6\rho$ and $b_3\delta + b_6\varepsilon = 0 = b_3\alpha + b_6\beta$ we get $a_4 = 0 = a_6$ and $b_3 = 0 = b_6$.

So we obtain in all these cases $H_{n',n'}g \cap gH_{n,n} = \{a_1 g \mid a_1 \in k^\times\}$. We conclude $|H_{n',n'}g \cap gH_{n,n}| = 1$ and when we use Lemma 3.5.12 3. we see that there are

$$\frac{|\Upsilon_{n',n',n,n}|}{|H_{n',n'}||H_{n,n}|} = \frac{(q-1)^2(q^2-1)^2(q^2+q+1)q^{2d}}{(q^2-1)(q^2-q)q^{2n'+2}(q^2-1)(q^2-q)q^{2n+2}} = (q^2+q+1)q^{2d-2n'-2n-6}$$

double cosets for $d \geq n' + n + 3 \geq 5$ (cf. 1(w)i and 2(w)i).

c) $n' > m' = 0$: The degree restraints for the entries of the matrix M are as follows:

$$deg(\alpha) \leq \kappa + n',\ deg(\beta) \leq \kappa + n',\ deg(\gamma) \leq \kappa + n + n',\ deg(\delta) \leq \kappa,\ deg(\varepsilon) \leq \kappa,\ deg(\theta) \leq \kappa + n,\ deg(\vartheta) \leq \kappa,\ deg(\rho) \leq \kappa,\ deg(\iota) \leq \kappa + n.$$

By multiplication with matrices from $H_{n',0}$ from the left and matrices from $H_{n,n}$ from the right we multiply the lower left 2×2 block submatrix $\tilde{M} := \begin{pmatrix} \delta & \varepsilon \\ \vartheta & \rho \end{pmatrix}$ of M with some element in $\mathrm{PGL}_2(K)$ from the left and from the right. This induces a multiplication of the determinant of $\tilde{M}$ with a non-zero element from the field k. So the degree of the determinant does not change by this multiplication. This is the reason for the following cases:

- If $det(\tilde{M})$ has not its maximal possible degree, then we choose the representative $g := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix}$ for the double coset of M, where the only entries of their respective maximal possible degree are those on the diagonal of g and furthermore, if $\vartheta \neq 0$ we choose g such that $\delta = 0$ or the equation $a\delta = b\vartheta$ with $a \in k$ and $b \in k[t]$ is only possible for $a = 0$.
- If $det(\tilde{M})$ has its maximal possible degree, we take the representative

$g := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix}$ with ϑ, ε and γ of their respective maximal possible degree and all other entries have less degree than it is possible. Furthermore we may choose g such that $deg(\alpha) < deg(\beta) < \kappa$ (multiply with a suitable matrix from $H_{n',0}$ if necessary, in particular we can multiply successive with the matrices $\begin{pmatrix} 1 & -\beta_{deg(\beta)}t^{deg(\beta)-\kappa} & -\alpha_{deg(\alpha)}t^{deg(\alpha)-\kappa} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ to decrease the respective degree of α and β in every step by 1. By these multiplications we end with $deg(\alpha) \leq deg(\beta) < \kappa$. Then we can use suitable matrices from $H_{n,n}$ and $H_{n',0}$ to get $deg(\alpha) < deg(\beta) < \kappa$ and we still have that only the diagonal entries are polynomials of their respective maximal possible degree).

We want to compute the intersection between

$$H_{n',0}g = \left\{ \begin{pmatrix} a_1\alpha + a_2\delta + a_3\vartheta & a_1\beta + a_2\varepsilon + a_3\rho & a_1\gamma + a_2\theta + a_3\iota \\ a_4\delta + a_5\vartheta & a_4\varepsilon + a_5\rho & a_4\theta + a_5\iota \\ a_6\delta + a_7\vartheta & a_6\varepsilon + a_7\rho & a_6\theta + a_7\iota \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & a_6 & a_7 \end{pmatrix} \in H_{n',0} \right\}$$

and

$$gH_{n,n} = \left\{ \begin{pmatrix} b_1\alpha + b_4\beta & b_2\alpha + b_5\beta & b_3\alpha + b_6\beta + b_7\gamma \\ b_1\delta + b_4\varepsilon & b_2\delta + b_5\varepsilon & b_3\delta + b_6\varepsilon + b_7\theta \\ b_1\vartheta + b_4\rho & b_2\vartheta + b_5\rho & b_3\vartheta + b_6\rho + b_7\iota \end{pmatrix} \mid \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ 0 & 0 & b_7 \end{pmatrix} \in H_{n,n} \right\}.$$

- Suppose $det(\tilde{M})$ is not maximal. By $a_4\delta + a_5\vartheta = b_1\delta + b_4\varepsilon$ we obtain $b_4 = 0$ and $(a_4 - b_1)\delta + a_5\vartheta = 0$. From $a_1\alpha + a_2\delta + a_3\vartheta = b_1\alpha + b_4\beta$ we conclude $a_1 = b_1$ and $a_2\delta + a_3\vartheta = 0$. Using $a_6\varepsilon + a_7\rho = b_2\vartheta + b_5\rho$ we find $a_6 = 0$ and $(a_7 - b_5)\rho = b_2\vartheta$. According to $a_4\varepsilon + a_5\rho = b_2\delta + b_5\varepsilon$ we get $a_4 = b_5$ and $a_5\rho = b_2\delta$. Moreover, the equation $a_6\theta + a_7\iota = b_3\vartheta + b_6\rho + b_7\iota$ yields $a_7 = b_7$ and $0 = b_3\vartheta + b_6\rho$.

  For $\vartheta = 0$ we have due to the determinant of g and the Irreducibility of f that $\delta \neq 0 \neq \rho$ and hence we know $a_4 = b_1$, $a_2 = 0$, $a_7 = b_5$ and $b_6 = 0$ from the equations $(a_4 - b_1)\delta = 0 = a_2\delta = (a_7 - b_5)\delta = b_6\rho$. The remaining equations are $a_5\rho = b_2\delta$, $a_3\rho = b_2\alpha$, $a_5\iota = b_3\delta$ and $a_3\iota = b_3\alpha$. Since α and δ are coprime polynomials we deduce that ι has to divide b_3, but $deg(\iota) = \kappa + n > n \geq deg(b_3)$. Hence $b_3 = 0$, which implies $a_5 = 0 = a_3 = b_2$.

For $\vartheta \neq 0$ we find $a_7 = b_1$, since $(a_7 - b_1)\vartheta = 0$. Now we distinguish between the cases $\delta = 0$ and $\delta \neq 0$:

We start with the case $\delta = 0$. Now $a_5\vartheta = 0 = a_3\vartheta$ yields $a_5 = 0 = a_3$. Moreover, we know that ε and θ are coprime polynomials and the degree of ε is equal to κ. Together with $(a_4 - b_7)\theta = b_6\varepsilon$ we derive $a_4 = b_7$ and $b_6 = 0$. With $b_2\vartheta = 0$ we find $b_2 = 0$. By $a_2\varepsilon = 0$ it is $a_2 = 0$ and hence $b_3\alpha = 0$, which yields $b_3 = 0$.

Now let δ be non-zero. According to $(a_4 - b_1)\delta + a_5\vartheta = 0$ and our choice of g we conclude $a_4 = b_1$ and $a_5 = 0$. Whence $b_2\delta = 0$, i.e. $b_2 = 0$. With Lemma 3.5.7 a) and the equations $a_2\delta + a_3\vartheta = 0 = a_2\varepsilon + a_3\rho$ and $b_3\vartheta + b_6\rho = 0 = b_3\alpha + b_6\beta$ we deduce $a_2 = 0 = a_3$ and $b_3 = 0 = b_6$.

- Suppose $det(\tilde{M})$ has maximal possible degree. From the equation $a_6\delta + a_7\vartheta = b_1\vartheta + b_4\rho$ we derive $a_7 = b_1$ and $a_6\delta = b_4\rho$ in this case. We use $a_4\varepsilon + a_5\rho = b_2\delta + b_5\varepsilon$ to see that $a_4 = b_5$ and $a_5\rho = b_2\delta$. The equation $a_1\gamma + a_2\theta + a_3\iota = b_3\alpha + b_6\beta + b_7\gamma$ yields $a_1 = b_7$ and $a_2\theta + a_3\iota = b_3\alpha + b_6\beta$. Next we consider the two equations $(a_1 - b_1)\alpha + a_2\delta + a_3\vartheta = b_4\beta$ and $(a_1 - b_5)\beta + a_2\varepsilon + a_3\rho = b_2\alpha$. Since we assume $deg(\alpha) < deg(\beta) < \kappa$ the first equation implies $deg(a_2) > deg(a_3)$ or $a_3 = 0$, but the second equation is only true for $deg(a_3) > deg(a_2)$ or $a_2 = 0$. Therefore we conclude that $a_2 = 0 = a_3$. Hence we have $(a_1 - b_1)\alpha = b_4\beta$ and $(a_1 - b_5)\beta = b_2\alpha$. Since $deg(\beta) > deg(\alpha)$ we find $b_4 = 0$, $(a_1 - b_1)\alpha = 0$ and $a_1 = b_5$, $b_2\alpha = 0$. Using $(a_4 - b_1)\delta + a_5\vartheta = 0$ it follows $a_5 = 0$ and $(a_4 - b_1)\delta = 0$. Applying Lemma 3.5.7 a) to $b_3\delta + b_6\varepsilon = 0 = b_3\alpha + b_6\beta$ yields $b_3 = 0 = b_6$.

  For $\delta = 0$ we have necessarily $\alpha \neq 0$ and hence $(a_1 - b_1)\alpha = 0$ and $b_2\alpha = 0$ imply $a_1 = b_1$ and $b_2 = 0$. Moreover, we have $\varepsilon \neq 0$, which implies $a_6 = 0$, since the equation $a_6\varepsilon = 0$ is remaining.

  For $\delta \neq 0$ the equations $b_2\delta = a_6\delta = 0 = (a_4 - b_1)\delta$ yield $b_2 = 0 = a_6$ and $a_4 = b_1$.

In all these cases we computed that the intersection $H_{n',0}g \cap gH_{n,n} = \{a_1 g \mid a_1 \in k^\times\}$ is trivial. According to Lemma 3.5.12 3. we get $\frac{|\Upsilon_{n',0,n,n}|}{|H_{n',0}||H_{n,n}|} = \frac{(q-1)^2(q^2-1)^2(q^2+q+1)q^{2d}}{(q^2-1)(q^2-q)q^{2n'+2}(q^2-1)(q^2-q)q^{2n+2}} = (q^2+q+1)q^{2d-2n'-2n-6}$ double cosets in this case, if $d \geq n' + n + 3 \geq 5$ (cf. 1(w)ii and 2(w)ii).

d) $n' = m' = 0$: In this case the degree restraints for the entries of M are given by

$$deg(\alpha) \leq \kappa,\ deg(\beta) \leq \kappa,\ deg(\gamma) \leq \kappa + n,\ deg(\delta) \leq \kappa,\ deg(\varepsilon) \leq \kappa,\ deg(\theta) \leq \kappa + n,\ deg(\vartheta) \leq \kappa,\ deg(\rho) \leq \kappa,\ deg(\iota) \leq \kappa + n.$$

By multiplication with suitable matrices from $H_{0,0}$ and $H_{n,n}$ we may take the representative $g := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix}$, where only the entries on the diagonal of g have their respective maximal possible degree and furthermore, we may choose $deg(\rho) > deg(\vartheta)$.

We calculate the intersection of

$$H_{0,0}g = \left\{ \begin{pmatrix} a_1\alpha + a_2\delta + a_3\vartheta & a_1\beta + a_2\varepsilon + a_3\rho & a_1\gamma + a_2\theta + a_3\iota \\ a_4\alpha + a_5\delta + a_6\vartheta & a_4\beta + a_5\varepsilon + a_6\rho & a_4\gamma + a_5\theta + a_6\iota \\ a_7\alpha + a_8\delta + a_9\vartheta & a_7\beta + a_8\varepsilon + a_9\rho & a_7\gamma + a_8\theta + a_9\iota \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} \in H_{0,0} \right\}$$

with

$$gH_{n,n} = \left\{ \begin{pmatrix} b_1\alpha + b_4\beta & b_2\alpha + b_5\beta & b_3\alpha + b_6\beta + b_7\gamma \\ b_1\delta + b_4\varepsilon & b_2\delta + b_5\varepsilon & b_3\delta + b_6\varepsilon + b_7\theta \\ b_1\vartheta + b_4\rho & b_2\vartheta + b_5\rho & b_3\vartheta + b_6\rho + b_7\iota \end{pmatrix} \mid \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ 0 & 0 & b_7 \end{pmatrix} \in H_{n,n} \right\}.$$

In the intersection the equation $a_7\alpha + a_8\delta + a_9\vartheta = b_1\vartheta + b_4\rho$ holds, which implies $a_7 = 0$ and $a_8\delta + a_9\vartheta = b_1\vartheta + b_4\rho$. According to $a_1\alpha + a_2\delta + a_3\vartheta = b_1\alpha + b_4\beta$ we have $a_1 = b_1$ and $a_2\delta + a_3\vartheta = b_4\beta$. Using $a_4\beta + a_5\varepsilon + a_6\rho = b_2\delta + b_5\varepsilon$ we find $a_5 = b_5$ and $a_4\beta + a_6\rho = b_2\delta$. With $a_7\beta + a_8\varepsilon + a_9\rho = b_2\vartheta + b_5\rho$ we see that $a_8 = 0$ and $(a_9 - b_5)\rho = b_2\vartheta$, which implies $a_9 = b_5$ and $b_2\vartheta = 0$. From $a_7\gamma + a_8\theta + a_9\iota = b_3\vartheta + b_6\rho + b_7\iota$ we deduce $a_9 = b_7$ and $0 = b_3\vartheta + b_6\rho$. Now we use the equation $(a_9 - b_1)\vartheta = b_4\rho$ to conclude that $b_4 = 0$ and $(a_9 - b_1)\vartheta = 0$. Insert this into $a_4\alpha + a_5\delta + a_6\vartheta = b_1\delta + b_4\varepsilon$ we get $a_4 = 0$ and $(a_5 - b_1)\delta + a_6\vartheta = 0$.

Suppose $\vartheta = 0$. By the determinant of g and the Irreducibility of f we know that δ and ρ necessarily have to be non-zero. Therefore the equations $(a_5 - b_1)\delta = 0 = b_6\rho = a_2\delta$ imply $a_5 = b_1$ and $a_2 = 0 = b_6$. Hence there are the following remaining equations: $a_3\rho = b_2\alpha$, $a_6\rho = b_2\delta$, $a_6\iota = b_3\delta$ and $a_3\iota = b_3\alpha$. Since the polynomials α and δ are coprime we deduce that ι divides b_3 and according to $deg(\iota) = \kappa + n > n \geq deg(b_3)$ we can conclude $b_3 = 0$. Now the remaining equations above yield $a_3 = 0 = a_6 = b_2$.

Suppose $\vartheta \neq 0$. Then we have $(a_9 - b_1)\vartheta = 0 = b_2\vartheta$, i.e. $a_9 = b_1$ and $b_2 = 0$. Next the equation $a_6\rho = 0$ is left, which means $a_6 = 0$. When we apply Lemma 3.5.7 a) to the equations $a_2\delta + a_3\vartheta = 0 = a_2\varepsilon + a_3\rho$ and $b_3\vartheta + b_6\rho = 0 = b_3\delta + b_6\varepsilon$ we find $a_2 = 0 = a_3$ and $b_3 = 0 = b_6$.

Therefore the intersection is $H_{0,0}g \cap gH_{n,n} = \{a_1 g \mid a_1 \in k^\times\}$, which implies $|H_{0,0}g \cap gH_{n,n}| = 1$. By Lemma 3.5.12 3. we conclude that there are

$$\frac{|\Upsilon_{0,0,n,n}|}{|H_{0,0}||H_{n,n}|} = \frac{(q-1)^2(q^2-1)^2(q^2+q+1)q^{2d}}{(q^3-1)(q^3-q)q^2(q^2-1)(q^2-q)q^{2n+2}} = q^{2d-2n-6}$$

double cosets for $d \geq n+3 \geq 4$ (cf. 1(g)i and 2(g)i).

3. $n > m = 0$:

   a) $n' > m' > 0$: For this case we use the symmetry 3.7.4 and the case $n > m > 0$ and $n' = m' > 0$ to obtain
   $\frac{|\Upsilon_{n',m',n,0}|}{|H_{n',m'}||H_{n,0}|} = \frac{(q-1)^2(q^2-1)^2(q^2+q+1)q^{2d}}{(q-1)^2q^{2n'+3}(q^2-1)(q^2-q)q^{2n+2}} = (q+1)(q^2+q+1)q^{2d-2n'-2n-6}$
   double cosets, if $d \geq n'+n+3 \geq 6$ (cf. 1(v)iv and 2(v)iv).

   b) $n' = m' > 0$: We have the following degree restraints for the entries of M:

   $$deg(\alpha) \leq \kappa+n',\ deg(\beta) \leq \kappa+n+n',\ deg(\gamma) \leq \kappa+n+n',\ deg(\delta) \leq \kappa+n',\ deg(\varepsilon) \leq \kappa+n+n',\ deg(\theta) \leq \kappa+n+n',\ deg(\vartheta) \leq \kappa,\ deg(\rho) \leq \kappa+n,\ deg(\iota) \leq \kappa+n.$$

   If we multiply the matrix M with matrices from $H_{n',n'}$ or $H_{n,0}$, then the polynomial ϑ is multiplied with non-zero elements from the field. This means we do not change the degree of ϑ via this multiplication. Hence we consider the following two cases:

   - If ϑ has not its maximal possible degree we choose as representative for the double coset the matrix $g := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix}$, where the only entries with their respective maximal possible degree are those on the diagonal of g. Furthermore, if $\vartheta \neq 0$ we choose g in such a way that $\delta = 0$ or the equation $a\delta = b\vartheta$ for $a \in k$ and $b \in k[t]$ implies $a = 0$. Additionally we do the same for ρ instead of δ, i.e. $\rho = 0$ or $a\rho = b\vartheta$ for $a \in k$ and $b \in k[t]$ implies $a = 0$.
   - If ϑ has its maximal possible degree we choose the representative $g := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix}$, such that ϑ, ε and γ are the only entries of g with their respective maximal possible degree. Moreover, we choose $\alpha = 0$ or the equation $a\alpha + b\delta + c\vartheta = 0$ with $a, b \in k$, $c \in k[t]$ implies $a = 0$, if this holds, we additionally choose $\delta = 0$ or the equation $a\delta = b\vartheta$ for $a \in k$, $b \in k[t]$ implies $a = 0$.

   We want to compute the intersection of

   $$H_{n',n'}g = \left\{ \begin{pmatrix} a_1\alpha + a_2\delta + a_3\vartheta & a_1\beta + a_2\varepsilon + a_3\rho & a_1\gamma + a_2\theta + a_3\iota \\ a_4\alpha + a_5\delta + a_6\vartheta & a_4\beta + a_5\varepsilon + a_6\rho & a_4\gamma + a_5\theta + a_6\iota \\ a_7\vartheta & a_7\rho & a_7\iota \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & a_7 \end{pmatrix} \in H_{n',n'} \right\}$$

with

$$gH_{n,0} =$$
$$\left\{ \begin{pmatrix} b_1\alpha & b_2\alpha + b_4\beta + b_6\gamma & b_3\alpha + b_5\beta + b_7\gamma \\ b_1\delta & b_2\delta + b_4\varepsilon + b_6\theta & b_3\delta + b_5\varepsilon + b_7\theta \\ b_1\vartheta & b_2\vartheta + b_4\rho + b_6\iota & b_3\vartheta + b_5\rho + b_7\iota \end{pmatrix} \mid \begin{pmatrix} b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 \\ 0 & b_6 & b_7 \end{pmatrix} \in H_{n,0} \right\}.$$

- Consider the case where ϑ has not its maximal possible degree. With $a_4\alpha + a_5\delta + a_6\vartheta = b_1\delta$ it follows $a_4 = 0$ and $(a_5 - b_1)\delta + a_6\vartheta = 0$. The equation $a_1\alpha + a_2\delta + a_3\vartheta = b_1\alpha$ yields $a_1 = b_1$ and $a_2\delta + a_3\vartheta = 0$. From $b_2\vartheta + b_4\rho + b_6\iota = a_7\rho$ we know that $b_6 = 0$ and $b_2\vartheta + (b_4 - a_7)\rho = 0$. Using $a_4\beta + a_5\varepsilon + a_6\rho = b_2\delta + b_4\varepsilon + b_6\theta$ we conclude $a_5 = b_4$ and $a_6\rho = b_2\delta$. Furthermore the equation $a_7\iota = b_3\vartheta + b_5\rho + b_7\iota$ implies $a_7 = b_7$ and $0 = b_3\vartheta + b_5\rho$.

  Suppose $\vartheta = 0$. This means that δ and ρ have to be non-zero since f is irreducible and hence the equations $(a_5 - b_1)\delta = a_2\delta = 0 = (b_4 - a_7)\rho = b_5\rho$ imply $a_5 = b_1$, $a_2 = 0$, $b_4 = a_7$ and $b_5 = 0$. The remaining equations are given by $a_6\rho = b_2\delta$, $a_3\rho = b_2\alpha$, $a_6\iota = b_3\delta$ and $a_3\iota = b_3\alpha$. Since ι and ρ are coprime polynomials it follows from these equations that α has to divide the polynomial a_3, but we know additionally $deg(\alpha) = \kappa + n' > n' \geq deg(a_3)$. Therefore we obtain $a_3 = 0$ and whence $b_2 = 0 = a_6 = b_3$.

  Suppose $\vartheta \neq 0$. By $(a_7 - b_1)\vartheta = 0$ we have $a_7 = b_1$.

  For $\delta = 0 = \rho$ we have $a_3 = 0 = a_6$ and $b_3 = 0 = b_6$, since in this case we have the equations where these entries multiplied with the non-zero element ϑ are equal to zero. Moreover, we know that β and ε are coprime polynomials with $deg(\varepsilon) > 0$ and hence we conclude from $(a_1 - b_4)\beta + a_2\varepsilon = 0$ that $a_1 = b_4$ and $a_2 = 0$.

  For $\delta \neq 0$ the equation $(a_5 - b_1)\delta + a_6\vartheta = 0$ implies $a_5 = b_1$ and $a_6 = 0$ because of our choice of g. This yields the equation $b_2\delta = 0$ and whence $b_2 = 0$. We apply Lemma 3.5.7 a) to $a_2\delta + a_3\vartheta = 0 = a_2\varepsilon + a_3\rho$ and $b_3\vartheta + b_5\rho = 0 = b_3\delta + b_5\varepsilon$ to derive $a_2 = 0 = a_3$ and $b_3 = 0 = b_5$.

  For $\rho \neq 0$ we use $b_2\vartheta + (b_4 - a_7)\rho = 0$ to obtain $b_2 = 0$ and $b_4 = a_7$ according to our choice of g. Similar to the case $\delta \neq 0$ we have now $a_6\rho = 0$, i.e. $a_6 = 0$. Next we apply again Lemma 3.5.7 a) to $a_2\delta + a_3\vartheta = 0 = a_2\varepsilon + a_3\rho$ and $b_3\vartheta + b_5\rho = 0 = b_3\delta + b_5\varepsilon$ to derive $a_2 = 0 = a_3$ and $b_3 = 0 = b_5$.

- If ϑ has its maximal possible degree we deduce from $(a_7 - b_1)\vartheta = 0$ that $a_7 = b_1$. With $a_4\beta + a_5\varepsilon + a_6\rho = b_2\delta + b_4\varepsilon + b_6\theta$ we have $a_5 = b_4$ and $a_4\beta + a_6\rho = b_2\delta + b_6\theta$. Moreover, the equation $a_1\gamma + a_2\theta + a_3\iota = b_3\alpha + b_5\beta + b_7\gamma$ implies $a_1 = b_7$ and $a_2\theta + a_3\iota = b_3\alpha + b_5\beta$.

For $\alpha = 0$ we have necessarily that δ and $\vartheta \in k[t] \setminus k$ are coprime polynomials. Therefore the equation $(a_5 - b_1)\delta = a_6\vartheta$ yields $a_5 = b_1$ and $a_6 = 0$. Because of $a_2\delta + a_3\vartheta = 0$ and ϑ has its maximal possible degree we obtain $a_2 = 0 = a_3$. Using this we have the equation $(a_1 - b_4)\beta = b_6\gamma$ left. Together with the fact that β and γ are coprime polynomials and γ has its maximal possible degree we deduce $a_1 = b_4$ and $b_6 = 0$. Next we can use $0 = b_5\beta$ to get $b_5 = 0$ and whence $b_3\vartheta = 0$ implies $b_3 = 0$. From $a_4\gamma = 0$ we deduce $a_4 = 0$ and therefore $b_2\delta = 0$, which means $b_2 = 0$.

For $\alpha \neq 0$ we know from our choice of g and $a_4\alpha + a_5\delta + a_6\vartheta = b_1\delta$ that $a_4 = 0$ and $(a_5 - b_1)\delta + a_6\vartheta = 0$. Furthermore, by the same argument $a_1\alpha + a_2\delta + a_3\vartheta = b_1\alpha$ yields $a_1 = b_1$ and $a_2\delta + a_3\vartheta = 0$.

If $\delta = 0$ we have $a_6\vartheta = 0 = a_3\vartheta$, in particular $a_6 = 0 = a_3$. In this case we know that ε and θ are coprime polynomials and ε is of its maximal possible degree. Hence $b_6\theta = 0$ and $(a_5 - b_7)\theta = b_5\varepsilon$ imply $b_6 = 0$ and $a_5 = b_7$, $b_5 = 0$. Some of the remaining equations are $b_3\vartheta = 0 = b_2\vartheta$, which means $b_3 = 0 = b_2$. Now it remains the equation $a_2\varepsilon = 0$, i.e. we obtain $a_2 = 0$.

If $\delta \neq 0$, then $(a_5 - b_1)\delta + a_6\vartheta = 0$ implies $a_5 = b_1$ and $a_6 = 0$ due to our choice of g. Similar, $a_2\delta + a_3\vartheta = 0$ yields $a_2 = 0 = a_3$. According to Lemma 3.5.7 a) the equations $b_2\vartheta + b_6\iota = 0 = b_2\delta + b_6\theta$ and $b_3\vartheta + b_5\rho = 0 = b_3\alpha + b_5\beta$ imply $b_2 = 0 = b_6$ and $b_3 = 0 = b_5$.

In all these cases we found $H_{n',n'}g \cap gH_{n,0} = \{a_1 g \mid a_1 \in k^\times\}$. This means $|H_{n',n'}g \cap gH_{n,0}| = 1$ and with Lemma 3.5.12 3. we compute

$$\frac{|\Upsilon_{n',n',n,0}|}{|H_{n',n'}||H_{n,0}|} = \frac{(q-1)^2(q^2-1)^2(q^2+q+1)q^{2d}}{(q^2-1)(q^2-q)q^{2n'+2}(q^2-1)(q^2-q)q^{2n+2}} = (q^2+q+1)q^{2d-2n'-2n-6}.$$

So this is the number of double cosets in this case, if the degree satisfies $d \geq n' + n + 3 \geq 5$ (cf. 1(w)iii and 2(w)iii).

c) $n' > m' = 0$: Using the symmetry 3.7.4 we compute this case from the case $n = m > 0$ and $n' = m' > 0$. Therefore we get

$$\frac{|\Upsilon_{n',0,n,0}|}{|H_{n',0}||H_{n,0}|} = \frac{(q-1)^2(q^2-1)^2(q^2+q+1)q^{2d}}{(q^2-1)(q^2-q)q^{2n'+2}(q^2-1)(q^2-q)q^{2n+2}} = (q^2+q+1)q^{2d-2n'-2n-6}$$

double cosets for $d \geq n' + n + 3 \geq 5$ (cf. 1(w)iv and 2(w)iv).

d) $n' = 0 = m'$: The degree restraint in this case are $deg(\alpha) \leq \kappa$, $deg(\beta) \leq \kappa + n$, $deg(\gamma) \leq \kappa + n$, $deg(\delta) \leq \kappa$, $deg(\varepsilon) \leq \kappa + n$, $deg(\theta) \leq \kappa + n$, $deg(\vartheta) \leq \kappa$, $deg(\rho) \leq \kappa + n$, $deg(\iota) \leq \kappa + n$.

As representative for the double coset of M we choose the matrix $g := \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \theta \\ \vartheta & \rho & \iota \end{pmatrix}$, where only the entries on the diagonal have their respective maximal possible degree. Furthermore, we may choose $deg(\delta) > deg(\vartheta)$.

Next we want to compute the intersection between

$$H_{0,0}g = \left\{ \begin{pmatrix} a_1\alpha + a_2\delta + a_3\vartheta & a_1\beta + a_2\varepsilon + a_3\rho & a_1\gamma + a_2\theta + a_3\iota \\ a_4\alpha + a_5\delta + a_6\vartheta & a_4\beta + a_5\varepsilon + a_6\rho & a_4\gamma + a_5\theta + a_6\iota \\ a_7\alpha + a_8\delta + a_9\vartheta & a_7\beta + a_8\varepsilon + a_9\rho & a_7\gamma + a_8\theta + a_9\iota \end{pmatrix} \mid \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} \in H_{0,0} \right\}$$

and

$$gH_{n,0} = \left\{ \begin{pmatrix} b_1\alpha & b_2\alpha + b_4\beta + b_6\gamma & b_3\alpha + b_5\beta + b_7\gamma \\ b_1\delta & b_2\delta + b_4\varepsilon + b_6\theta & b_3\delta + b_5\varepsilon + b_7\theta \\ b_1\vartheta & b_2\vartheta + b_4\rho + b_6\iota & b_3\vartheta + b_5\rho + b_7\iota \end{pmatrix} \mid \begin{pmatrix} b_1 & b_2 & b_3 \\ 0 & b_4 & b_5 \\ 0 & b_6 & b_7 \end{pmatrix} \in H_{n,0} \right\}.$$

In this intersection we have the equation $a_7\alpha+a_8\delta+a_9\vartheta = b_1\vartheta$, which implies $a_7 = 0 = a_8$ and $(a_9-b_1)\vartheta = 0$. By $a_4\alpha+a_5\delta+a_6\vartheta = b_1\delta$ it follows $a_4 = 0$ and $(a_5-b_1)\delta+a_6\vartheta = 0$, i.e. $a_5 = b_1$ and $a_6\vartheta = 0$. With $a_1\alpha+a_2\delta+a_3\vartheta = b_1\alpha$ we conclude that $a_1 = b_1$ and $a_2\delta + a_3\vartheta = 0$, in particular $a_2 = 0$ and $a_3\vartheta = 0$. From $a_9\rho = b_2\vartheta + b_4\rho + b_6\iota$ we deduce $b_6 = 0$ and $(a_9 - b_4)\rho = b_2\vartheta$. Using $a_4\beta + a_5\varepsilon + a_6\rho = b_2\delta + b_4\varepsilon + b_6\theta$ we know $a_5 = b_4$ and $a_6\rho = b_2\delta$. The equation $a_7\gamma + a_8\theta + a_9\iota = b_3\vartheta + b_5\rho + b_7\iota$ yields $a_9 = b_7$ and $0 = b_3\vartheta + b_5\rho$.

Suppose $\vartheta = 0$, then $\rho \neq 0$ because f is irreducible and hence $0 = b_3\vartheta + b_5\rho$ implies $b_5 = 0$ and $(a_9 - b_4)\rho = b_2\vartheta$ yields $a_9 = b_4$. Now we use the equation $a_4\gamma + a_5\theta + a_6\iota = b_3\delta + b_5\varepsilon + b_7\theta$ to derive $a_6 = 0$ and $b_3\delta = 0$, in particular $b_3 = 0$, since $\delta \neq 0$. Similar, $b_2\delta = 0$ implies $b_2 = 0$. Hence, we get $a_3 = 0$ from $a_3\rho = 0$.

Suppose $\vartheta \neq 0$, this implies $a_3 = 0 = a_6$ and $a_9 = b_1$. Thus $b_2\delta = 0$, i.e. since $deg(\delta) > deg(\vartheta)$ this means $b_2 = 0$. According to Lemma 3.5.7 a) the equations $b_3\delta + b_5\varepsilon = 0 = b_3\vartheta + b_5\rho$ imply $b_3 = 0 = b_5$.

This proves that $H_{0,0}g \cap gH_{n,0} = \{a_1 g \mid a_1 \in k^\times\}$ is trivial. Due to Lemma 3.5.12 3. we calculate

$$\frac{|\Upsilon_{0,0,n,0}|}{|H_{0,0}||H_{n,0}|} = \frac{(q-1)^2(q^2-1)^2(q^2+q+1)q^{2d}}{(q^3-1)(q^3-q)q^2(q^2-1)(q^2-q)q^{2n+2}} = q^{2d-2n-6}$$

double cosets for this case, if $d \geq n+3 \geq 4$ (cf. 1(g)iii and 2(g)iii).

4. $n = 0 = m$:

   a) $n' > m' > 0$: This case follows by symmetry 3.7.4 from the case $n > m > 0$ and $n' = 0 = m'$. Hence we have

   $$\frac{|\Upsilon_{n',m',0,0}|}{|H_{n',m'}||H_{0,0}|} = \frac{(q-1)^2(q^2-1)^2(q^2+q+1)q^{2d}}{(q-1)^2q^{2n'+3}(q^3-1)(q^3-q)q^2} = (q+1)q^{2d-2n'-6}$$

   double cosets for $d \geq n'+3 \geq 5$ (cf. 1(n)ii and 2(n)ii).

b) $n' = m' > 0$: We use the symmetry 3.7.4 and the case $n > m = 0$ and $n' = 0 = m'$ to obtain
$$\frac{|\Upsilon_{n',n',0,0}|}{|H_{n',n'}||H_{0,0}|} = \frac{(q-1)^2(q^2-1)^2(q^2+q+1)q^{2d}}{(q^2-1)(q^2-q)q^{2n'+2}(q^3-1)(q^3-q)q^2} = q^{2d-2n'-6}$$
double cosets for this case, if the degree fulfills $d \geq n' + 3 \geq 4$ (cf. 1(g)iv and 2(g)iv).

c) $n' > m' = 0$: With the case $n = m > 0$ and $n' = 0 = m'$ we can use the symmetry 3.7.4 to solve this case. As solution we get
$$\frac{|\Upsilon_{n',0,0,0}|}{|H_{n',0}||H_{0,0}|} = \frac{(q-1)^2(q^2-1)^2(q^2+q+1)q^{2d}}{(q^2-1)(q^2-q)q^{2n'+2}(q^3-1)(q^3-q)q^2} = q^{2d-2n'-6}$$
double cosets for $d \geq n' + 3 \geq 4$ (cf. 1(g)ii and 2(g)ii).

d) $n' = 0 = m'$: Note that we have necessarily 3 divides d to get a solution for this case.

Similar to Subcase 3.c in [KMS15] we start to count all orbits of edges containing the vertex $x_{0,0}$ under its vertex stabilizer $\tilde{\Gamma}_{x_{0,0}} = \Xi_{y_{0,0}} = \mathrm{PGL}_3(k)$ (see Proposition 3.4.8). Instead of these orbits we can consider the $\mathrm{PGL}_3(k)$-orbits of points and lines in the projective plane $\mathbb{P}_2(\mathbb{F}_{q^d})$ (due to Remark 2.6.3).

The action of $\mathrm{PGL}_3(k)$ on the projective plane $\mathbb{P}_2(\mathbb{F}_{q^d})$: The projective plane is given by
$$\mathbb{P}_2(\mathbb{F}_{q^d}) = \left\{ \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \mid x, y \in \mathbb{F}_{q^d} \right\} \cup \left\{ \begin{pmatrix} x \\ 1 \\ 0 \end{pmatrix} \mid x \in \mathbb{F}_{q^d} \right\} \cup \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Let $A := \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} \in \mathrm{GL}_3(k)$.

If A stabilizes a point $\mathfrak{p}$ in the projective plane $\mathbb{P}_2(\mathbb{F}_{q^d})$ this means that there exists some $\zeta \in \mathbb{F}_{q^d}^\times$, such that the equation $A\mathfrak{p} = \zeta\mathfrak{p}$ is satisfied. This implies that ζ has to be an eigenvalue of the matrix A. Hence, ζ is a root of the characteristic polynomial of A, in particular ζ is a root of a polynomial of degree at most 3. This implies that ζ has to be in the field extension $\mathbb{F}_{q^2}$ or $\mathbb{F}_{q^3}$ of the field $\mathbb{F}_q$. Notice that we always have $\mathbb{F}_{q^3} \subseteq \mathbb{F}_{q^d}$, because 3 divides d, but $\mathbb{F}_{q^2}$ is a subset of $\mathbb{F}_{q^d}$ if and only if d is even. So we have $\mathbb{P}_2(\mathbb{F}_{q^3}) \subseteq \mathbb{P}_2(\mathbb{F}_{q^d})$. This is the reason why we first consider the action of $\mathrm{PGL}_3(k)$ on the projective plane $\mathbb{P}_2(\mathbb{F}_{q^3})$:

For $\mathfrak{p} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$: If A stabilizes the point $\mathfrak{p}$ we have $A\mathfrak{p} = \zeta\mathfrak{p}$ for some

$\zeta \in \mathbb{F}_{q^3}^\times$. This is equivalent to $\begin{pmatrix} a_1 \\ a_4 \\ a_7 \end{pmatrix} = \zeta \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Hence $a_4 = 0 = a_7$ and $a_1 = \zeta \in k^\times$. Since A is a matrix in $\mathrm{GL}_3(k)$ it follows $\begin{pmatrix} a_5 & a_6 \\ a_8 & a_9 \end{pmatrix} \in \mathrm{GL}_2(k)$ and $a_2, a_3 \in k$. Therefore the stabilizer in $\mathrm{PGL}_3(k)$ of the point $\mathfrak{p}$ has cardinality $|\mathrm{PGL}_3(k)_\mathfrak{p}| = \frac{q-1}{q-1}(q^2-1)(q^2-q)q^2$. We divide by $q-1$ to take into account that we are working in the projective group. Due to the orbit-stabilizer theorem we have

$$|\mathrm{PGL}_3(k)\mathfrak{p}| = \frac{|\mathrm{PGL}_3(k)|}{|\mathrm{PGL}_3(k)_\mathfrak{p}|} = \frac{(q^3-1)(q^3-q)q^2}{(q^2-1)(q^2-q)q^2} = \frac{q^3-1}{q-1} = q^2+q+1.$$

This corresponds to the embedding $\mathbb{P}_2(\mathbb{F}_q) \subseteq \mathbb{P}_2(\mathbb{F}_{q^d})$.

Now we choose $z \in \mathbb{F}_{q^3}$ such that $(1, z, z^2)$ is a basis of $\mathbb{F}_{q^3}$ over k.

For $\mathfrak{p} = \begin{pmatrix} z \\ 1 \\ 0 \end{pmatrix}$: The equation $A\mathfrak{p} = \zeta\mathfrak{p}$ is equal to $\begin{pmatrix} a_1 z + a_2 \\ a_4 z + a_5 \\ a_7 z + a_8 \end{pmatrix} = \zeta \begin{pmatrix} z \\ 1 \\ 0 \end{pmatrix}$. Therefore $a_7 = 0 = a_8$, $a_4 z + a_5 = \zeta$ and $a_1 z + a_2 = \zeta z$, whence $a_1 z + a_2 = a_4 z^2 + a_5 z$. Since we took $z \in \mathbb{F}_{q^3} \setminus \mathbb{F}_q$ it can not be a solution of a quadratic equation, in particular, we can conclude from the equation that $a_4 = 0 = a_2$ and $a_1 = a_5 \in k$, to be precise $a_1 = a_5 \in k^\times$, because $A \in \mathrm{GL}_3(k)$. For the last column of A we have to take $a_9 \in k^\times$ and $a_3, a_6 \in k$. We conclude $|\mathrm{PGL}_3(k)_\mathfrak{p}| = \frac{q-1}{q-1}(q-1)q^2$. According to the orbit-stabilizer theorem we compute

$$|\mathrm{PGL}_3(k)\mathfrak{p}| = \frac{|\mathrm{PGL}_3(k)|}{|\mathrm{PGL}_3(k)_\mathfrak{p}|} = \frac{(q^3-1)(q^3-q)q^2}{(q-1)q^2} = \frac{(q^3-1)(q^3-q)}{q-1} = (q^3-q)(q^2+q+1).$$

For $\mathfrak{p} = \begin{pmatrix} z^2 \\ z \\ 1 \end{pmatrix}$: We know that $\mathrm{GL}_3(k)$ acts transitively on the set of bases of $\mathbb{F}_{q^3}$. So we have $\mathrm{PGL}_3(k)\mathfrak{p} = \left\{ \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \mid (x, y, 1) \text{ is a basis of } \mathbb{F}_{q^3} \right\}$ and hence $|\mathrm{PGL}_3(k)\mathfrak{p}| = (q^3-q)(q^3-q^2)$.

Because of $q^2+q+1+(q^3-q)(q^2+q+1)+(q^3-q)(q^3-q^2) = q^6+q^3+1 = |\mathbb{P}_2(\mathbb{F}_{q^3})|$ we see that $\mathrm{PGL}_3(k)$ acts with three orbits on the projective plane $\mathbb{P}_2(\mathbb{F}_{q^3})$.

Next we have to distinguish between odd and even degree d.

If d is odd, then $\mathbb{F}_{q^2}$ is not a subfield of $\mathbb{F}_{q^d}$.

For $\mathfrak{p} = \begin{pmatrix} a \\ 1 \\ 0 \end{pmatrix}$, where $a \in \mathbb{F}_{q^d} \setminus \mathbb{F}_{q^3}$: Analogously to the case $\mathfrak{p} = \begin{pmatrix} z \\ 1 \\ 0 \end{pmatrix}$ above, we may compute

$|\mathrm{PGL}_3(k)\mathfrak{p}| = \frac{|\mathrm{PGL}_3(k)|}{|\mathrm{PGL}_3(k)_{\mathfrak{p}}|} = \frac{(q^3-1)(q^3-q)q^2}{(q-1)q^2} = \frac{(q^3-1)(q^3-q)}{q-1} = (q^3-q)(q^2+q+1)$.

The points in the projective plane who lie in one of these orbits are the points

$$\left\{ \begin{pmatrix} y \\ x \\ 1 \end{pmatrix} \mid y \in \mathbb{F}_{q^d} \setminus \mathbb{F}_{q^3}, x \in \langle 1, y\rangle_k \setminus \mathbb{F}_q \right\} \cup \left\{ \begin{pmatrix} y \\ x \\ 1 \end{pmatrix} \mid y \in \mathbb{F}_{q^d} \setminus \mathbb{F}_{q^3}, x \in \mathbb{F}_q \right\}$$

$\cup \left\{ \begin{pmatrix} y \\ x \\ 1 \end{pmatrix} \mid y \in \mathbb{F}_q, x \in \mathbb{F}_{q^d} \setminus \mathbb{F}_{q^3} \right\} \cup \left\{ \begin{pmatrix} y \\ 1 \\ 0 \end{pmatrix} \mid y \in \mathbb{F}_{q^d} \setminus \mathbb{F}_{q^3} \right\}$. Therefore we find $(q^d - q^3)(q^2 - q) + 2(q^d - q^3)q + q^d - q^3 = (q^d - q^3)(q^2 + q + 1)$ points of the projective plane who lie in an orbit of this kind.

This implies

$$\frac{(q^d-q^3)(q^2+q+1)}{(q^3-q)(q^2+q+1)} = q^{d-3} + q^{d-5} + \ldots + q^2$$

orbits of this form.

All the other points in the projective plane $\mathbb{P}_2(\mathbb{F}_{q^d})$ have trivial stabilizers in $\mathrm{PGL}_3(k)$. So the cardinality of such an orbit is given by the cardinality of the group $\mathrm{PGL}_3(k)$, hence it equals $(q^3 - 1)(q^3 - q)q^2$. We have $|\mathbb{P}_2(\mathbb{F}_{q^d})| = q^{2d} + q^d + 1$. Thus there are $q^{2d} + q^d + 1 - (q^6 + q^3 + 1) - (q^d - q^3)(q^2 + q + 1) = q^{2d} - q^6 - (q^d - q^3)(q^2 + q) = q^{2d} - q^{d+2} - q^{d+1} - q^6 + q^5 + q^4$ points in the projective plane with trivial stabilizer. Whence, we obtain

$$\frac{q^{2d}-q^{d+2}-q^{d+1}-q^6+q^5+q^4}{(q^3-1)(q^3-q)q^2} = \frac{q^{2d}-q^{d+2}-q^{d+1}-q^6+q^5+q^4}{q^8-q^6-q^5+q^3}$$

orbits.

In total we have 3 orbits for $d = 3$ and $3+q^{d-3}+q^{d-5}+\ldots+q^2+\frac{q^{2d}-q^{d+2}-q^{d+1}-q^6+q^5+q^4}{q^8-q^6-q^5+q^3}$ orbits for the case $d \geq 9$ is odd.

If d is even, then $\mathbb{F}_{q^2}$ is a subfield of $\mathbb{F}_{q^d}$ and hence the projective plane over $\mathbb{F}_{q^2}$ a subset of the projective plane over $\mathbb{F}_{q^d}$. Therefore we have to consider the action of $\mathrm{PGL}_3(k)$ on the projective plane over $\mathbb{F}_{q^2}$: Let $v \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. Note that $(1, v)$ is a basis of $\mathbb{F}_{q^2}$ over k.

For $\mathfrak{p} = \begin{pmatrix} v \\ 1 \\ 0 \end{pmatrix}$: The equation $A\mathfrak{p} = \zeta\mathfrak{p}$ means $\begin{pmatrix} a_1v + a_2 \\ a_4v + a_5 \\ a_7v + a_8 \end{pmatrix} = \zeta \begin{pmatrix} v \\ 1 \\ 0 \end{pmatrix}$. We deduce $a_7 = 0 = a_8$, $a_4v + a_5 = \zeta$ and $a_1v + a_2 = \zeta v$. Thus $a_1v + a_2 = (a_4v + a_5)v$. Moreover, we know $a_9 \in k^\times$, since $A \in \mathrm{GL}_3(k)$. And for the same reason we have $\begin{pmatrix} a_1 & a_2 \\ a_4 & a_5 \end{pmatrix} \in \mathrm{GL}_2(k)$. The last two entries of A are arbitrary elements in the field k. From the arguments in [KMS15] we conclude that $|\mathrm{PGL}_2(k)_{\binom{v}{1}}| = q+1$. Now we have to multiply this cardinality with the possibilities for the last column of A, in particular we multiply with $(q-1)q^2$. Then we obtain $|\mathrm{PGL}_3(k)_{\mathfrak{p}}| = (q+1)(q-1)q^2 = (q^2-1)q^2$. By the orbit-stabilizer theorem we know $|\mathrm{PGL}_3(k)\mathfrak{p}| = \frac{|\mathrm{PGL}_3(k)|}{|\mathrm{PGL}_3(k)_{\mathfrak{p}}|} = \frac{(q^3-1)(q^3-q)q^2}{(q+1)(q-1)q^2} = (q^3-1)q$. With $|\mathbb{P}_2(\mathbb{F}_{q^2})\backslash\mathbb{P}_2(\mathbb{F}_q)| = q^4+q^2+1-(q^2+q+1) = q^4-q = (q^3-1)q$ we find one orbit for the action of $\mathrm{PGL}_3(k)$ on the projective plane over $\mathbb{F}_{q^2}$.

For $\mathfrak{p} = \begin{pmatrix} b \\ 1 \\ 0 \end{pmatrix}$ with $b \in \mathbb{F}_{q^d} \setminus (\mathbb{F}_{q^2} \cup \mathbb{F}_{q^3})$: Similar to the above cases of this form we calculate $|\mathrm{PGL}_3(k)_{\mathfrak{p}}| = (q-1)q^2$ and hence $|\mathrm{PGL}_3(k)\mathfrak{p}| = (q^3-q)(q^2+q+1)$. Points in the projective plane $\mathbb{P}_2(\mathbb{F}_{q^d})$ who lie in one of these orbits are points from the following set:

$$\left\{ \begin{pmatrix} y \\ x \\ 1 \end{pmatrix} \mid y \in \mathbb{F}_{q^d} \setminus (\mathbb{F}_{q^2} \cup \mathbb{F}_{q^3}), x \in \langle 1, y \rangle_k \setminus \mathbb{F}_q \right\} \cup$$
$$\left\{ \begin{pmatrix} y \\ x \\ 1 \end{pmatrix} \mid y \in \mathbb{F}_{q^d} \setminus (\mathbb{F}_{q^2} \cup \mathbb{F}_{q^3}), x \in \mathbb{F}_q \right\} \cup$$
$$\left\{ \begin{pmatrix} y \\ x \\ 1 \end{pmatrix} \mid y \in \mathbb{F}_q, x \in \mathbb{F}_{q^d} \setminus (\mathbb{F}_{q^2} \cup \mathbb{F}_{q^3}) \right\} \cup \left\{ \begin{pmatrix} y \\ 1 \\ 0 \end{pmatrix} \mid y \in \mathbb{F}_{q^d} \setminus (\mathbb{F}_{q^2} \cup \mathbb{F}_{q^3}) \right\}.$$

With $|\mathbb{F}_{q^2} \cup \mathbb{F}_{q^3}| = |\mathbb{F}_{q^2}| + |\mathbb{F}_{q^3}| - |\mathbb{F}_{q^2} \cap \mathbb{F}_{q^3}| = |\mathbb{F}_{q^2}| + |\mathbb{F}_{q^3}| - |\mathbb{F}_q| = q^2+q^3-q$ we find $|\mathbb{F}_{q^d} \setminus (\mathbb{F}_{q^2} \cup \mathbb{F}_{q^3})| = q^d-q^3-q^2+q$. This implies we have $(q^d-q^3-q^2+q)(q^2-q)+2(q^d-q^3-q^2+q)q+q^d-q^3-q^2+q = (q^d-q^3-q^2+q)(q^2+q+1)$ points which are in one of these orbits. So we obtain

$$\tfrac{(q^d-q^3-q^2+q)(q^2+q+1)}{(q^3-q)(q^2+q+1)} = q^{d-3} + q^{d-5} + ... + q^3 + q - 1$$

orbits of this form.

All the other points in $\mathbb{P}_2(\mathbb{F}_{q^d})$ have trivial stabilizer and hence the length of the corresponding orbit is maximal, in particular equals the cardinality of $\mathrm{PGL}_3(k)$. The number of remaining points in the projective plane over $\mathbb{F}_{q^d}$ is $q^{2d}+q^d+1-(q^6+q^3+1)-(q^4+q^2+1)+(q^2+q+1)-(q^d-q^3-q^2+q)(q^2+q+1) = q^{2d}-q^{d+2}-q^{d+1}-q^6+q^5+q^4$. Therefore we have

$$\tfrac{q^{2d}-q^{d+2}-q^{d+1}-q^6+q^5+q^4}{q^8-q^6-q^5+q^3}$$

orbits of maximal length.

In total we have

$$3 + 1 + q^{d-3} + q^{d-5} + ... + q^3 + q - 1 + \tfrac{q^{2d}-q^{d+2}-q^{d+1}-q^6+q^5+q^4}{q^8-q^6-q^5+q^3}$$

orbits for $d \geq 6$ is even.

The action of $\mathrm{PGL}_3(k)$ on the set of two dimensional subspaces of $\mathbb{F}^3_{q^d}$: Each two dimensional subspace of $\mathbb{F}^3_{q^d}$ is the kernel of a linear map, which can be represented by a vector in the dual space of $\mathbb{F}^3_{q^d}$, in particular there exists a vector (x, y, z) in the dual space of $\mathbb{F}^3_{q^d}$, such that the points in the two dimensional subspace are those points $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{F}^3_{q^d}$ where the equation $(x, y, z)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = xa + yb + zc = 0$ holds. Now we can identify the dual space of $\mathbb{F}^3_{q^d}$ with $\mathbb{F}^3_{q^d}$ itself to obtain a bijective correspondence between the two dimensional subspaces of $\mathbb{F}^3_{q^d}$ with the points in the projective plane $\mathbb{P}_2(\mathbb{F}_{q^d})$. Let $A \in \mathrm{PGL}_3(k)$ and E be a two dimensional subspace of $\mathbb{F}^3_{q^d}$. If E is represented by the vector (x, y, z) in the dual space of $\mathbb{F}^3_{q^d}$, then the two dimensional subspace AE is represented by $(x, y, z)A^{-1}$. Identifying the dual space of $\mathbb{F}^3_{q^d}$ with $\mathbb{F}^3_{q^d}$ itself, yields that the two dimensional subspace AE is represented by the point $(A^{-1})^T\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ of the projective plane $\mathbb{P}_2(\mathbb{F}_{q^d})$. Since $\phi : \mathrm{PGL}_3(k) \to \mathrm{PGL}_3(k), A \mapsto (A^{-1})^T$ is a bijection, the action of $\mathrm{PGL}_3(k)$ on the two dimensional subspaces of $\mathbb{F}^3_{q^d}$ is given by the action of $\mathrm{PGL}_3(k)$ on the points of the projective plane $\mathbb{P}_2(\mathbb{F}_{q^d})$. So we obtain again the number of orbits we computed from the action of $\mathrm{PGL}_3(k)$ on $\mathbb{P}_2(\mathbb{F}_{q^d})$ as result for the action of $\mathrm{PGL}_3(k)$ on the two dimensional subspaces of $\mathbb{F}^3_{q^d}$.

This leads to a total amount of 6 orbits for $d = 3$,

$$6 + 2(q^{d-3} + q^{d-5} + ... + q^2) + 2\tfrac{q^{2d}-q^{d+2}-q^{d+1}-q^6+q^5+q^4}{q^8-q^6-q^5+q^3}$$

orbits for the case $d \geq 9$ is odd and

$$6 + 2(q^{d-3} + q^{d-5} + ... + q^3 + q) + 2\tfrac{q^{2d}-q^{d+2}-q^{d+1}-q^6+q^5+q^4}{q^8-q^6-q^5+q^3}$$

orbits for $d \geq 6$ is even. For each of these orbits we have a corresponding edge in the quotient $\tilde{\Gamma}\backslash X$, which contains the vertex $X_{0,0}$.

In order to compute the number of edges from $X_{0,0}$ to itself we have to subtract from the total amount of edges containing $X_{0,0}$ the number of all edges from

$X_{0,0}$ to other vertices. We calculated this number of edges in previous cases: For d is odd we already have

$$1+q^2+q^4+q^6+...+q^{d-5}+q^{2(d-\frac{d+1}{2}-1)}= \\ 1+q^2+q^4+q^6+...+q^{d-5}+q^{d-3}=\tfrac{q^{d-1}-1}{q^2-1}$$

edges for the case $\kappa=0$, $n>m>0$, $n'=m'=0$;
1 edge for the case $\kappa=0$, $n=m>0$, $n'=m'=0$;
1 edge for the case $\kappa=0$, $n=m=0$, $n'>m'=0$;

$$1+q^2+q^4+...+q^{d-5}+q^{2(\frac{d-1}{2}-1)}=1+q^2+q^4+...+q^{d-5}+q^{d-3}=\tfrac{q^{d-1}-1}{q^2-1}$$

edges for the case $\kappa=0$, $n=m=0$, $n'>m'>0$;

$$q^{2d-12}+q^{2d-18}+...+q^6+q^{2d-2(d-3)-6}=q^{2d-12}+q^{2d-18}+...+q^6+1= \\ \tfrac{q^{2d-6}-1}{q^6-1}=\tfrac{(q^{d-3}+1)(q^{d-3}-1)}{(q^3+1)(q^3-1)}$$

edges for the case $\kappa=\frac{d-n'}{3}\in\mathbb{N}$, $n=m=0$, $n'>m'=0$ (note that $\kappa=\frac{d-n'}{3}\in\mathbb{N}$ implies $n'\in 3\mathbb{N}$ and $d-3\geq n'\geq 3$);

$$q^{2d-12}+q^{2d-18}+...+q^{d+3}+q^{2d-2\frac{d-3}{2}-6}= \\ q^{2d-12}+q^{2d-18}+...+q^{d+3}+q^{d-3}=q^{d-3}\tfrac{q^{d-3}-1}{q^6-1}$$

edges for the case $\kappa=\frac{d-2n'}{3}\in\mathbb{N}$, $n=m=0$, $n'=m'>0$ (note that $\kappa=\frac{d-2n'}{3}\in\mathbb{N}$ implies $n'\in 3\mathbb{N}$ and $\frac{d-3}{2}\geq n'\geq 3$);
the number of edges in the case $\kappa=\frac{d-n'-m'}{3}\in\mathbb{N}$, $n=m=0$, $n'>m'>0$ is equal to

$$(q+1)\left(\sum_{m'=1}^{\frac{d-5}{2}}\sum_{i=0}^{l(m')}q^{2m'+6i}\right),$$

with $l(m')=\begin{cases}\frac{d-2m'-4}{3} & \text{if } m'\equiv 1 \pmod 3\\ \frac{d-2m'-5}{3} & \text{if } m'\equiv 2 \pmod 3\\ \frac{d-2m'-6}{3} & \text{if } m'\equiv 0 \pmod 3\end{cases}$. (Note that $\kappa=\frac{d-n'-m'}{3}\in\mathbb{N}$ implies $n'+m'\in 3\mathbb{N}$ and $\frac{d-5}{2}\geq m'\geq 1$; moreover $\frac{d-5}{2}\equiv -1 \pmod 3$). The number of edges in each of the last three cases comes twice. This is because of the symmetry 3.7.4 we get the same amount of edges for the cases $n=m>0$, $n'=m'=0$; $n>m=0$, $n'=m'=0$ and $n>m>0$, $n'=m'=0$, respectively.

For d is even we already have

$$1+q^2+q^4+q^6+...+q^{d-5}+q^{2(d-(\frac{d}{2}+1)-1)}= \\ 1+q^2+q^4+q^6+...+q^{d-6}+q^{d-4}=\tfrac{q^{d-2}-1}{q^2-1}$$

edges for the case $\kappa = 0$, $n > m > 0$, $n' = m' = 0$;
1 edge for the case $\kappa = 0$, $n = m > 0$, $n' = m' = 0$;
1 edge for the case $\kappa = 0$, $n = m = 0$, $n' > m' = 0$;

$$1 + q^2 + q^4 + ... + q^{d-5} + q^{2(\frac{d}{2}-1-1)} = 1 + q^2 + q^4 + ... + q^{d-6} + q^{d-4} = \frac{q^{d-2}-1}{q^2-1}$$

edges for the case $\kappa = 0$, $n = m = 0$, $n' > m' > 0$;
moreover we found

$$\frac{q(q^{d-3}+1)}{q+1} = q^{d-3} - q^{d-4} + q^{d-5} - ... + q^3 - q^2 + q$$

edges for the case $\kappa = 0$, $n > m = 0$, $n' = m' = 0$;

$$\frac{q(q^{d-3}+1)}{q+1} = q^{d-3} - q^{d-4} + q^{d-5} - ... + q^3 - q^2 + q$$

edges for the case $\kappa = 0$, $n = m = 0$, $n' = m' > 0$;

$$q^{2d-12} + q^{2d-18} + ... + q^6 + q^{2d-2(d-3)-6} = q^{2d-12} + q^{2d-18} + ... + q^6 + 1 = \frac{q^{2d-6}-1}{q^6-1} = \frac{(q^{d-3}+1)(q^{d-3}-1)}{(q^3+1)(q^3-1)}$$

edges for the case $\kappa = \frac{d-n'}{3} \in \mathbb{N}$, $n = m = 0$, $n' > m' = 0$ (note that $\kappa = \frac{d-n'}{3} \in \mathbb{N}$ implies $n' \in 3\mathbb{N}$ and $d - 3 \geq n' \geq 3$);

$$q^{2d-12} + q^{2d-18} + ... + q^{d+6} + q^{2d-2(\frac{d}{2}-3)-6} = q^{2d-12} + q^{2d-18} + ... + q^{d+6} + q^d = q^d \frac{q^{d-6}-1}{q^6-1}$$

edges for the case $\kappa = \frac{d-2n'}{3} \in \mathbb{N}$, $n = m = 0$, $n' = m' > 0$ (note that $\kappa = \frac{d-2n'}{3} \in \mathbb{N}$ implies $n' \in 3\mathbb{N}$ and $\frac{d}{2} - 3 = \frac{d-6}{2} \geq n' \geq 3$);
the number of edges in the case $\kappa = \frac{d-n'-m'}{3} \in \mathbb{N}$, $n = m = 0$, $n' > m' > 0$ is equal to

$$(q+1)\left(\sum_{m'=1}^{\frac{d-4}{2}} \sum_{i=0}^{l(m')} q^{2m'+6i}\right),$$

with $l(m') = \begin{cases} \frac{d-2m'-4}{3} & \text{if } m' \equiv 1 \pmod 3 \\ \frac{d-2m'-5}{3} & \text{if } m' \equiv 2 \pmod 3 \\ \frac{d-2m'-6}{3} & \text{if } m' \equiv 0 \pmod 3 \end{cases}$. (Note that $\kappa = \frac{d-n'-m'}{3} \in \mathbb{N}$ implies $n' + m' \in 3\mathbb{N}$ and $\frac{d-4}{2} \geq m' \geq 1$; moreover $\frac{d-4}{2} \equiv 1 \pmod 3$). The number of edges in each of the last three cases comes twice. The reason is because we get by symmetry 3.7.4 the same amount of edges for the cases $n = m > 0$, $n' = m' = 0$; $n > m = 0$, $n' = m' = 0$ and $n > m > 0$, $n' = m' = 0$, respectively.

Therefore we find from $X_{0,0}$ to itself in total 2 edges for $d = 3$,

$$2+2\frac{q^{2d}-q^{d+2}-q^{d+1}-q^6+q^5+q^4}{q^8-q^6-q^5+q^3}-2(q^{2d-12}+q^{2d-18}+...+q^6+1)-2(q^{2d-12}+ q^{2d-18}+...+q^{d+3}+q^{d-3})-2(q+1)\left(\sum_{m'=1}^{\frac{d-5}{2}}\sum_{i=0}^{l(m')}q^{2m'+6i}\right)$$

edges for $d \geq 9$ is odd and

$$2+2\frac{q^{2d}-q^{d+2}-q^{d+1}-q^6+q^5+q^4}{q^8-q^6-q^5+q^3}-2(q^{2d-12}+q^{2d-18}+...+q^6+1)-2(q^{2d-12}+ q^{2d-18}+...+q^{d+6}+q^{d})-2(q+1)\left(\sum_{m'=1}^{\frac{d-4}{2}}\sum_{i=0}^{l(m')}q^{2m'+6i}\right)$$

edges for $d \geq 6$ is even. Since we have three different types of vertices we have to divide this number by 2 to get the number of edges between the vertices corresponding to $X_{0,0}$ (cf. 2z).

This proves our Main Theorem 3.1.1. Notice that we have to distinguish whether the degree d is divisible by 3 or not. If 3 divides d, then we have three different types of vertices since Γ preserves types, whereas $\tilde{\Gamma}$ does not preserve the types of the vertices. In case 3 does not divide d we have $\Gamma = \tilde{\Gamma}$ and hence $\Gamma\backslash X = \tilde{\Gamma}\backslash X$.

A. Appendix

A.1. The PGL_2 case

We consider the situation in [KMS15]. With the methods we used to solve the PGL_3 case we can give another proof for the PGL_2 case, which was solved in [KMS15]. We use the notation from [KMS15]. As first step we compute the cardinality of the set $\Upsilon_{n,m} = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mid \begin{matrix} \alpha, \beta, \gamma, \delta \in k[t]; deg(\alpha) \leq \frac{d+n-m}{2}, deg(\beta) \leq \frac{d+n+m}{2}, \\ deg(\gamma) \leq \frac{d-n-m}{2}, deg(\delta) \leq \frac{d-n+m}{2}; \alpha\delta - \gamma\beta = \lambda f, \lambda \in k^{\times} \end{matrix} \right\}$. We do this similar to Lemma 3.5.12: Thus we count the possibilities for choosing a matrix $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Upsilon_{n,m}$. First we have to choose the first column of M in such a way that α and γ are coprime polynomials and at least one of them has its maximal possible degree. This is done in [KMS15]. Hence there are $(q-1)^2(q^{d-m} + q^{d-m-1})$ possible choices for the first column of M. Remember $l = \frac{d-n-m}{2}$. Next we want to count the possible choices for the second column of M such that $det(M) = f$. Therefore we solve the system of linear equations $Ax = b$, where $x = \begin{pmatrix} \delta_0 \\ \vdots \\ \delta_{l+m} \\ \beta_0 \\ \vdots \\ \beta_{l+m+n} \end{pmatrix}$, $b = \begin{pmatrix} f_0 \\ \vdots \\ f_d \end{pmatrix}$ and

$$A = \begin{pmatrix} \alpha_0 & 0 & & 0 & \gamma_0 & 0 & & 0 \\ \vdots & \alpha_0 & \ddots & & \vdots & \gamma_0 & \ddots & \\ \alpha_{l+n} & \vdots & & & \gamma_l & \vdots & & \\ 0 & \alpha_{l+n} & \ddots & 0 & 0 & \gamma_l & \ddots & 0 \\ \vdots & 0 & \ddots & \alpha_0 & \vdots & 0 & \ddots & \gamma_0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \alpha_{l+n} & 0 & 0 & 0 & \gamma_l \end{pmatrix} \in k^{(d+1)\times(d+m+2)}.$$

Claim: The rank of A is maximal, in particular $rank(A) = d + 1$.

Proof: W. l. o. g. let $deg(\alpha) = l + n$. We consider the submatrix

$$B = \begin{pmatrix} \alpha_0 & 0 & & 0 & \gamma_0 & 0 & & 0 \\ \vdots & \alpha_0 & \ddots & & \vdots & \gamma_0 & \ddots & \\ \alpha_{l+n} & \vdots & & & \gamma_l & \vdots & & \\ 0 & \alpha_{l+n} & \ddots & 0 & 0 & \gamma_l & \ddots & 0 \\ \vdots & 0 & \ddots & \alpha_0 & \vdots & 0 & \ddots & \gamma_0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \alpha_{l+n} & 0 & 0 & 0 & \gamma_l \end{pmatrix} \in k^{(d-m)\times(d-m)},$$ where we erase the last $m+1$ columns of A, which have coefficients of α as entries and additionally, the last $m+1$ columns of A. Then the resulting matrix B has rank $d-m$: If we have a linear combination of the columns of B equal to zero, then we may write this equation as $\alpha g + \gamma h = 0$, with polynomials g and h, where $deg(g) \leq l-1$ and $deg(h) \leq l+n-1$. Since α and γ are coprime polynomials it follows that α divides h. But then $deg(\alpha) = l+n > l+n-1 \geq deg(h)$ implies $h=0$ and hence we also have $g=0$. Whence $rank(B) = d-m$.

Now we consider the submatrix of A consisting of B together with all columns, where the entries are coefficients of α. This matrix has rank $d+1$. Therefore we deduce $rank(A) = d+1$. For the system of equations $Ax = b$ we get $q^{d+m+2-(d+1)} = q^{m+1}$ solutions.

Since the determinant of M can be a scalar multiple of f we have $q-1$ choices for this scalar.

In total we have $(q-1)q^{m+1}$ choices for the second column of M.

Since we are working in the projective group we have to divide by $q-1$. So we obtain

$$|\Upsilon_{n,m}| = (q-1)^2(q^{d+1}+q^d).$$

It is also possible to use the methods for PGL_3 to compute the size of the double cosets in the last case $m+n<d$ in [KMS15]. The case $n,m>0$: We need to compute the intersection $H_n g \cap gH_m$. Therefore we may choose the matrix $g := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, such that $deg(\alpha) = l+n$ is the maximal possible degree, as representative for the double coset. Using this we calculate

$H_n g = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} g = \begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\gamma & c\delta \end{pmatrix} \mid a, c \in k^\times, b \in k[t] \text{ with } deg(b) \leq n \right\}$

and $gH_m = \left\{ g \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} = \begin{pmatrix} \alpha u & v\alpha + w\beta \\ \gamma u & v\gamma + w\delta \end{pmatrix} \mid u, w \in k^\times, v \in k[t] \text{ with } deg(v) \leq m \right\}$.

In the intersection $H_n g \cap gH_m$ we have the equations

$$c\gamma = u\gamma, (a-u)\alpha + b\gamma = 0, v\gamma + \delta(w-c) = 0, v\alpha + w\beta = a\beta + b\delta.$$

Since $\alpha\delta - \gamma\beta$ is a scalar multiple of the irreducible polynomial f it follows from the degree restraints that γ has to be non-zero. According to the above equations we deduce $c = u$. Moreover, γ and α have to be coprime polynomials. Hence we derive from the second equation that α has to divide b. But due to the degree restraints we know $deg(\alpha) = l + n > n \geq deg(b)$. This yields $b = 0$ and whence $a = u$. Now the last two equations are $v\gamma + \delta(w-u) = 0 = v\alpha + \beta(w-u)$, in particular $v\begin{pmatrix}\alpha\\ \gamma\end{pmatrix} + (w-u)\begin{pmatrix}\beta\\ \delta\end{pmatrix} = 0$. Because of $det(g) \neq 0$ we conclude $v = 0$ and $w = u$. It follows $H_n g \cap gH_m = \{ag \mid a \in k^\times\}$ and hence $|H_n g \cap gH_m| = \frac{q-1}{q-1} = 1$.

Similarly for the case $m > 0$ and $n = 0$: We may choose the matrix $g := \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$, such that $deg(\alpha) = l$ is the maximal possible degree, as representative for the double coset. With this representative we compute
$H_0 g = \left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix} g = \begin{pmatrix}a\alpha + b\gamma & a\beta + b\delta\\ c\alpha + d\gamma & c\beta + d\delta\end{pmatrix} \mid \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathrm{GL}_2(k)\right\}$
and $gH_m = \left\{g\begin{pmatrix}u & v\\ 0 & w\end{pmatrix} = \begin{pmatrix}\alpha u & v\alpha + w\beta\\ \gamma u & v\gamma + w\delta\end{pmatrix} \mid u, w \in k^\times, v \in k[t] \text{ with } deg(v) \leq m\right\}$.

In the intersection $H_0 g \cap gH_m$ we have the equations

$$c\alpha + d\gamma = u\gamma, (a-u)\alpha + b\gamma = 0, v\gamma + \delta(w-d) = c\beta, v\alpha + w\beta = a\beta + b\delta.$$

We have again $\gamma \neq 0$, because $\alpha\delta - \gamma\beta$ is a scalar multiple of the irreducible polynomial f. Due to the fact that γ and α have to be coprime polynomials we can use the second equation to find that α has to divide b. Again by the degree restraints we know $deg(\alpha) = l > 0 \geq deg(b)$. We conclude $b = 0$ and hence $a = u$. Analogously, the first equation yields $c = 0$ and $u = d$. The result is $H_n g \cap gH_m = \{ag \mid a \in k^\times\}$, i.e. $|H_n g \cap gH_m| = \frac{q-1}{q-1} = 1$.

Now we proved in both subcases that there is only one length for the double cosets. Therefore the number of double cosets is just $\frac{|\Upsilon_{n,m}|}{|H_m||H_n|}$.

To solve the last case, where all indices are equal to zero, we used the same arguments as given in the case $n = 0 = m$ in [KMS15].

A.2. S-Arithmetic Groups

Now we briefly give the definition of a S-arithmetic group. We will not go into details. There exist several books about algebraic geometry. The book [NX09] is one example. For more details about linear algebraic groups see, for instance, Chapter 2 in [Spr98]. The present section is based on Section 13.5 in [AB08] and Chapter 1 + Chapter 2 in [Spr98].

Let k be a an algebraically closed field. The closed sets in the Zariski topology on k^m are the solution sets of systems of polynomial equations. We can identify the $n \times n$

matrices over k with k^{n^2} and use this to define the Zariski topology on the set of matrices. Then we have the induced topology on the subset GL_n.

Definition A.2.1. A *linear algebraic group* is a closed subgroup of some GL_n with respect to the Zariski topology.

Let K be the function field of an irreducible, projective, smooth curve C defined over a finite field $k = \mathbb{F}_q$.

Example A.2.2. The rational function field $K = k(t)$ corresponds to the projective line.

Definition A.2.3. Denote by S a finite nonempty set of (closed) points of C and let $\mathcal{O}_S$ be the ring of functions in K that have no poles except possibly at points in S. In particular $\mathcal{O}_S = \bigcap_{p \notin S} \mathcal{O}_{\nu_p}$. Then $\mathcal{O}_S$ is called the ring of *S-integers*.

Definition A.2.4. Two Groups are said to be *commensurable* if their intersection is of finite index in both of them.

Definition A.2.5. Let G be a linear algebraic group defined over K. Then any subgroup of $G(K)$ commensurable with $G(\mathcal{O}_S)$ is called an *S-arithmetic group*.

Example A.2.6. We consider the group PGL_3 over the rational function field $K = k(t)$. Then for a place p of $k(t)$ the group $\mathrm{PGL}_3(\mathcal{O}_{\{p\}})$ is an S-arithmetic group with $S = \{p\}$.

A.3. Some open Questions

In this Thesis we focused on the action of the projective general linear group on the underlying graph of the associated Bruhat-Tits building. The Main Theorem 3.1.1 describes the quotient graphs of the underlying graph by the action of the arithmetic subgroup. So we do not consider what happens with the other simplices. In particular, we do not know what happens to the 2-simplices of the Bruhat-Tits building we act on. In one special case there is an answer to this question: For the case $\mathrm{PGL}_3(k[t]) \cong \mathrm{PGL}_3(\mathcal{O}_{\{\infty\}})$ the quotient of the building modulo the group action was computed by Soulè in [Sou79]. He proved that a sector is a simplicial fundamental domain, so when we insert for every triangle in the quotient graph for degree 1 a 2-simplex, then the result is a fundamental domain for the action on the Bruhat-Tits building.

For our proof we need the fact that $\mathcal{O}_{\{\infty\}} \cong k[t]$ is Euclidean. Therefore it is a further question what happens when the valuation ring $\mathcal{O}_{\{\infty\}}$ is not Euclidean.

Another further question is how the quotients look like for the actions of $\mathrm{PGL}_n(\mathcal{O}_{\{p\}})$, for $n \geq 4$, on the associated Bruhat-Tits buildings. Of course in higher dimension we have additionally the higher rank simplices, i.e. the simplices of rank r for $2 \leq r \leq n$. Thus another question is how the projective general linear group acts on the set of r-simplices, where $2 \leq r \leq n$.

Index

Bibliography

[AA93] Herbert Abels and Peter Abramenko. On the homotopy type of subcomplexes of Tits buildings. *Adv. Math.*, 101(1):78–86, 1993.

[AB87] Herbert Abels and Kenneth S. Brown. Finiteness properties of solvable S-arithmetic groups: an example. *J. Pure Appl. Algebra*, 44:77–83, 1987.

[AB08] Peter Abramenko and Kenneth S. Brown. *Buildings: theory and applications.* Springer, New York, Berlin, 2008.

[Abe91] Herbert Abels. Finiteness properties of certain arithmetic groups in the function field case. *Israel Jornal of Mathematics*, 76:113–128, 1991.

[Abr87] Peter Abramenko. *Endlichkeitseigenschaften der Gruppen* $SL_n(F_q[t])$. PhD thesis, Frankfurt am Main, 1987.

[Abr94] Peter Abramenko. Reduktive Gruppen über lokalen Körpern und Bruhat-Tits-Gebäude. *Bayreuth. Math. Schr.*, 47:1–69, 1994.

[Abr96] Peter Abramenko. *Twin buildings and applications to S-arithmetic groups.* Lecture notes in mathematics, Vol. 1641. Springer, Berlin, 1996.

[AN02] Peter Abramenko and Gabriele Nebe. Lattice chain models for affine buildings of classical type. *Math. Ann.*, 322(3):537–562, 2002.

[BB07] A. T. Benjamin and C. D. Bennett. The probability of relatively prime polynomials. *Mathematics Magazine*, 80(3):196–202, 2007.

[Beh98] Helmut Behr. Arithmetic groups over function fields. I. A complete characterization of finitely generated and finitely presented arithmetic subgroups of reductive algebraic groups. *Journal für die reine und angewandte Mathematik*, 495:79–118, 1998.

[BK15] Rony Bitan and R. Köhl. A building-theoretic approach to relative Tamagawa numbers of semisimple groups over global function fields. *Funct. Approx. Comment. Math.*, 53(2):215–247, 2015.

[BKW13] Kai-Uwe Bux, R. Köhl, and Stefan Witzel. Higher finiteness properties of reductive arithmetic groups in positive characteristic: the rank theorem. *Annals of Mathematics*, 177:311–366, 2013.

[Bou02] Nicolas Bourbaki. *Lie groups and Lie algebras. Chapters 4–6.* Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002. Translated from the 1968 French original by Andrew Pressley.

[Bro96] Kenneth S. Brown. *Buildings.* Springer, New York, Berlin, 1996.

[BS76] A. Borel and Jean-Pierre Serre. Cohomologie d'immeubles et de groupes S-arithmétiques. *Topology*, (15):211–232, 1976.

[BT72] F. Bruhat and J. Tits. Groupes réductifs sur un corps local. *Inst. Hautes Études Sci. Publ. Math.*, (41):5–48, 1972.

[BT84] F. Bruhat and J. Tits. Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée. *Inst. Hautes Études Sci. Publ. Math.*, (60):197–376, 1984.

[CR06] Charles W. Curtis and Irving Reiner. *Representation Theory of Finite Groups and Associative Algebras.* American Mathematical Society, Providence, Rhode Island, 2006.

[Gar97] Paul Garret. *Buildings and classical groups.* Chapman and Hall, London, 1997.

[Hum80] James E. Humphreys. *Arithmetic Groups.* Lecture Notes in Mathematics, Vol. 789. Springer-Verlag, Berlin-New York, 1980.

[Hum90] James E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1990.

[Hum98] James E. Humphreys. *Linear algebraic groups.* Graduate texts in mathematics, 21. Springer-Verlag, Berlin-New York, 1998.

[IM65] N. Iwahori and H. Matsumoto. On some Bruhat decomposition and the structure of the Hecke rings of $\mathfrak{p}$-adic Chevalley groups. *Inst. Hautes Études Sci. Publ. Math.*, (25):5–48, 1965.

[KMS15] R. Köhl, B. Mühlherr, and K. Struyve. Quotients of trees for arithmetic subgroups of PGL_2 over a rational function field. *Group theory*, 18(1):61–74, 2015.

[Lan96] Erasmus Landvogt. *A Compactification of the Bruhat-Tits Building*. Lecture Notes in Mathematics, Vol. 1619. Springer-Verlag, Berlin, 1996.

[Mor15] Dave Witte Morris. *Linear algebraic groups*. [Milton Keynes, UK] : Deductive Press, 2015. Version 1.0.

[MPW15] Bernhardt Mühlherr, Holger P. Petersson, and Richard M. Weiss. *Descent in Buildings*. Princeton, New York, 2015.

[NX09] Harald Niederreiter and Chaoping Xing. *Algebraic geometry in coding theory and cryptography*. Princeton, New York, 2009.

[Pap12] Athanase Papadopoulos. *Handbook of Teichmüller theory*. European Mathematical Society, Zürich, 2012. Volume 3.

[Pra77] Gopal Prasad. Strong approximation for semi-simple groups over function fields. *Ann. of Math.*, 105:553–572, 1977.

[Pra89] Gopal Prasad. Volumes of S-arithmetic quotients of semi-simple groups. *Publ. Math. IHES*, 69:91–114, 1989.

[Ron09] Mark Ronan. *Lectures on buildings*. University of Chicago Press, Chicago, IL, 2009. Updated and revised.

[Ser80] Jean-Pierre Serre. *Trees*. Springer, Berlin, 1980.

[Ser00] Jean-Pierre Serre. *Local fields*. Graduate texts in Mathematics, 67. Springer-Verlag, New York, 2000.

[Sou79] Christophe Soulè. Chevalley groups over polynomial rings. In *Homological group theory (C.T.C. Wall, ed.)*, volume 36 of *London Mathematical Society Lecture Note Series*, pages 359–367. Cambridge University Press, Cambridge, 1979.

[Spr98] Tonny A. Springer. *Linear algebraic groups*. Progress in mathematics 9. Birkhäuser, Boston, 1998. 2. ed.

[Stu80] Ulrich Stuhler. Homological properties of certain arithmetic groups in the function field case. *Inventiones Mathematicae*, (57):263–281, 1980.

[Tit74] Jacques Tits. *Buildings of spherical type and finite BN-pairs*. Lecture Notes in Mathematics, Vol. 386. Springer-Verlag, Berlin-New York, 1974.

[TW02] Jacques Tits and Richard M. Weiss. *Moufang Polygons*. Springer, Berlin, 2002.

[Wei03] Richard M. Weiss. *The structure of spherical buildings.* Princeton, Princeton, 2003.

[Wei09] Richard M. Weiss. *The structure of affine buildings.* Princeton, New York, 2009.

[Wer01] Annete Werner. Compactification of the Bruhat-Tits building of PGL by lattices of smaller rank. *Documenta Math.*, 6:315–342, 2001.

[Wer04] Annete Werner. Compactification of the Bruhat-Tits building of PGL by seminorms. *Math. Z.*, 248:511–526, 2004.

[Wer07] Annete Werner. Compactifications of Bruhat-Tits buildings associated to linear representations. *Proc. Lond. Math. Soc.*, 95:497–518, 2007.

[Wit14] Stefan Witzel. *Finiteness Properties of Arithmetic Groups Acting on Twin Buildings.* Lecture Notes in Mathematics, Vol. 2109. Springer, Berlin, 2014.

Selbstständigkeitserklärung

Ich erkläre: Ich habe die vorgelegte Dissertation selbständig und ohne unerlaubte fremde Hilfe und nur mit den Hilfen angefertigt, die ich in der Dissertation angegeben habe. Alle Textstellen, die wörtlich oder sinngemäß aus veröffentlichten Schriften entnommen sind, und alle Angaben, die auf mündlichen Auskünften beruhen, sind als solche kenntlich gemacht. Bei den von mir durchgeführten und in der Dissertation erwähnten Untersuchungen habe ich die Grundsätze guter wissenschaftlicher Praxis, wie sie in der „Satzung der Justus-Liebig-Universität Gießen zur Sicherung guter wissenschaftlicher Praxis" niedergelegt sind, eingehalten.

Gießen, April 2018